Fluctuating hydrodynamics of passive and active fluids

Chiu Fan Lee

Department of Bioengineering, Imperial College London

July 27, 2018

Contents

1 Preliminaries

1.1 Hydrodynamics

Hydrodynamics is the study of phenomena in the large distance and long time limits. As an example, consider the following equation of motion (**EOM**) of a scalar field $\phi(t, \mathbf{r})$ in d spatial dimenions:

$$\partial_t \phi + \beta \partial_t^2 \phi = \mu \nabla^2 \phi + \mu' \nabla^4 \phi \ . \tag{1}$$

By Fourier transforming the above EOM both temporally and spatially, i.e.,

$$\phi(\omega, \mathbf{k}) = \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}^d r \mathrm{e}^{\mathrm{i}(\omega - \mathbf{k}\cdot\mathbf{r})} \phi(t, \mathbf{r}) \ , \tag{2}$$

we arrive at

$$\mathrm{i}\omega\phi - \beta\omega^2\phi = -\mu k^2 \phi + \mu' k^4 \phi \ . \tag{3}$$

The hydrodynamic limit now refers to the limits of $\omega, k = |\mathbf{k}| \to 0$. As a result, the leading contributions in the EOM in this limit are terms linear in ω and quadratic in k. The hydrodynamic equation of the original EOM is thus simplify the diffusion equation

$$\partial_t \phi = \mu \nabla^2 \phi \ . \tag{4}$$

1.2 Fluctuations

Unless the temperature is at absolute zero, physical systems are always under thermal perturbations and thus fluctuating, e.g., Brownian motion. Let us use a random walk to illustrate how fluctuations can be modelled.

A diffusive random walk in $1d$ can be modelled by the following Langevin equation, which is a stochastic differential equation:

$$\mathrm{d}_t x = f(t) \tag{5}$$

where $f(t)$ is a Gaussian noise term with statistics

$$\langle f(t) \rangle = 0 \quad , \quad \langle f(t) f(t') \rangle = 2D\delta(t - t') \ , \tag{6}$$

where the angular brackets refer to the averaging over the noise.

Let $x(0) = 0$, we can solve the above equation by integrating:

$$x(t) = \int_0^t \mathrm{d}s f(s) \ . \tag{7}$$

In particular,

$$\langle x(t) \rangle = \int_0^t \mathrm{d}s \langle f(s) \rangle = 0 \tag{8}$$

$$\langle x(t)^2 \rangle = \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \langle f(s) f(s') \rangle = 2Dt \ . \tag{9}$$

We thus see the mean squared displacement goes linearly with t, which is exactly what we would expect from a random walk. We can also now see that D is the diffusion coefficient of this random walk.

Aside: How to simulate the Langevin equation (5). To integrate the SDE above numerically, we first pick a small time increment $\triangle t$, we can evolve the equation to the next time as follows:

$$x(t + \triangle t) = x(t) + \sqrt{2D\triangle t}\ r_t \ , \tag{10}$$

where r_t is random variable drawn from the normal distribution with zero mean and unit variance. Now for any $T = N\triangle t$, the above will give $x(T) = \sqrt{2D\triangle t} \sum_{k=1}^N r_t = \sqrt{2D\triangle t} R$, where R is a random variable of standard deviation $\sqrt{N}$. Therefore $\langle x(t)^2 \rangle = 2D\triangle t N = 2DT$, thus reproducing (9).

1.3 Summary

- Hydrodynamics = $\omega, k \to 0$.
- Fluctuations = adding noise terms in the EOM.

2 Hydrodynamics of passive (equilibrium) fluids

2.1 Equilibrium fluctuations

We have seen how fluctuations can be incorporated in the EOM, but it turns out that at thermal equilibrium, the strength of the noise term (D) is not arbitrary, instead, it is completely fixed once other parameters in the EOM are specified so that the system follows the Boltzmann distribution. To illustrate this, we will look at the case of a particle along a $1d$ track under thermal fluctuations in a quadratic potential well of the form $U(x) = Ax^2/2$.

From the Boltzmann postulate, we know that that the probability of finding the particles at position x is proportional to $\mathrm{e}^{-Ax^2/2k_BT}$. Therefore,

$$\langle x^2 \rangle = \frac{\int_{-\infty}^{\infty} \mathrm{d}x x^2 \mathrm{e}^{-Ax^2/2k_BT}}{\int_{-\infty}^{\infty} \mathrm{d}x \mathrm{e}^{-Ax^2/2k_BT}} = \frac{k_BT}{A} \tag{11}$$

If we describe the motion of the particle in an over-damped environment using a stochastic differential equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{\zeta}\partial_x U(x(t)) + f(t) \tag{12}$$

where ζ is the damping coefficient, and f is a Gaussian noise term of the form:

$$\langle f(t) \rangle = 0 \quad , \quad \langle f(t) f(t') \rangle = 2D\delta(t-t') \ . \tag{13}$$

The solution to (12) is

$$x(t) = \mathrm{e}^{-At/\zeta} x(0) + \mathrm{e}^{-At/\zeta} \int_0^t \mathrm{d}s \mathrm{e}^{sA/\zeta} f(s) \ . \tag{14}$$

From this, we can calculate the variance in x at $t \gg 0$:

$$\langle x(t)^2 \rangle = \mathrm{e}^{-2At/\zeta} \int_0^t \int_0^t \mathrm{d}s \mathrm{d}s' \mathrm{e}^{(s+s')A/\zeta} \langle f(s) f(s') \rangle \tag{15}$$

$$= \frac{\zeta D}{A} \ . \tag{16}$$

Comparing the above with (11), we have

$$D = \frac{k_BT}{\zeta} \ , \tag{17}$$

i.e., the noise strength D is completely fixed by the damping coefficient and temperature.

Re-writing it slightly as

$$\frac{D}{k_BT} = \frac{1}{\zeta} \ , \tag{18}$$

where the left-hand side is all about fluctuations while the right-hand side is about dissipation, this is thus called a *fluctuation-dissipation relation* (**FDR**), the first of which was discovered by Einstein in 1905.

2.2 Governing equation of passive fluids

Symmetry plays a central role in how we do modern physics. For an equilibrium system, symmetry constrains the allowable form of the Hamiltonian of the system [1]. For a non-equilibrium system, although a Hamiltonian may no longer be relevant, we can still use symmetry to deduce the form of EOM [2, 3]. To illustrate this approach, we will now review how such a symmetry consideration can help us derive the incompressible Navier-Stokes equation.

In an incompressible fluid, the obvious field variable is the velocity field $\mathbf{v}(\mathbf{r}, t)$, whose dynamics can be written as:

$$\partial_t \mathbf{v} = \frac{\mathbf{F}}{\rho} \ , \tag{19}$$

where ρ is the density field and $\mathbf{F}$ corresponds to the local force density. Since the system is incompressible, ρ is constant everywhere and we will ignore this constant factor from now on.

We now impose the following symmetries:

1. **Temporal invariance**: $\mathbf{F}$ does not depend on time t explicitly, hence forbidding terms like $t\mathbf{v}$. This symmetry means that experimental results on the fluid motion do not depend on the day of the week on which the experiments are done.

2. **Translational invariance**: $\mathbf{F}$ does not depend on the spatial location $\mathbf{r}$ explicitly, hence forbidding terms like $\mathbf{r}$. This symmetry means that experimental results do not depend on the location where the experiments are done

3. **Rotational invariance**: the EOM is invariance if the reference frame is rotated, hence forbidding terms like $\mathbf{w}$ for some constant vector $\mathbf{w}$. This symmetry means that experimental results do not depend on which direction the experimental apparatus are positioned towards.

4. **Parity invariance**: the EOM is invariant under spatial inversion, hence forbidding terms like $\nabla \times \mathbf{v}$. This symmetry means that the physical system has no chirality, i.e., the physics of fluid motion has no handedness.

Imposing these symmetries, and expanding $\mathbf{F}$ in powers of $\mathbf{v}$ and of the spatial derivatives ∇, we arrive at the generic EOM:

$$\partial_t \mathbf{v} = -\vec{\kappa} - \lambda(\mathbf{v} \cdot \nabla)\mathbf{v} + \mu\nabla^2\mathbf{v} + (a - bv^2)\mathbf{v} + \mu'(\nabla^2)^2\mathbf{v} + cv^4\mathbf{v} + \dots \tag{20}$$

where $v \equiv |\mathbf{v}|$ and "$\dots$" refer to higher order terms permissible in $\mathbf{F}$ that are not shown. Note that the first term on the R.H.S. of the above equation, $-\vec{\kappa}$, is a vectorial Lagrange multiplier there to enforce the incompressibility condition $\nabla \cdot \mathbf{v} = 0$. By the Helmholtz decomposition, we can write $\mathbf{F}$ as $\nabla p + \nabla \times \mathbf{A}$ where p is a scalar field and $\mathbf{A}$ is a vector field [4]. Since we want to subtract off part of $\mathbf{F}$ that is *not* divergence-free, we have $\vec{\kappa} = \nabla p$.

Our EOM so far does not look like the Navier-Stokes equation yet as we are still missing one crucial symmetry: the Galilean invariance.

5. **Galilean invariance**: when no external forces are acting on the system, the EOM is invariant if the reference frame is boosted to another reference frame that is travelling at a constant speed in an arbitrary direction.

Under this additional symmetry, the EOM remains invariant if we perform the following simultaneous transformations: $\mathbf{r} \mapsto \mathbf{r} - \mathbf{w}t$ and $\mathbf{v}(\mathbf{r}, t) \mapsto \mathbf{v}(\mathbf{r} - \mathbf{w}t, t) + \mathbf{w}$, for some arbitrary vector $\mathbf{w}$. Imposing this constraint, the EOM, to order $\mathcal{O}(\nabla^4)$, is

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} \ , \tag{21}$$

which is exactly the incompressible Navier-Stokes equation, with p interpreted as the pressure divided by the density.

In the hydrodynamic limit, higher order terms, such as $\nabla^4\mathbf{v}$, are unimportant compared to $\nabla^2\mathbf{v}$, and thus Eq. (21) can be viewed as the hydrodynamic equation of incompressible fluids.

2.3 What fluctuations to add?

We have thus far derived the Navier-Stokes equation in the incompressible limit, the next question is what are the fluctuation terms in the EOM? To find the appropriate fluctuation term, we rely on the FDR again. First, statistical mechanics, and specifically the equipartition theorem, says that

$$\langle v^2 \rangle = \frac{dk_BT}{\rho V} \tag{22}$$

where ρV is the mass of the system in a volume V that travels with speed v.

Now, what noise should we add to the system to recover this condition from our EOM? For fluids, the situation is a bit more complicated than diffusion since in a single-phase fluid, momentum conservation eliminates the presence of force monopolesa and thus the fluctuating forces has to be written as the spatial derivatives of some noise term that enters the EOM.

To proceed further, let us consider a simpler $1d$ fluid with the advective term, pressure term and density fluctuation ignored:

$$\partial_t v = \mu \partial_x^2 v + \partial_x f \ , \tag{23}$$

where the noise term f is again assumed to be of the form

$$\langle f(t,x) \rangle = 0 \quad , \quad \langle f(t,x) f(t',x') \rangle = 2D\delta(t-t')\delta(x-x') \ . \tag{24}$$

To solve equation (23), we Fourier transform v and f spatially to obtain [Exercise]

$$\partial_t v(t,k) = -\mu k^2 v(t,k) - \mathrm{i} k f(t,k) \tag{25}$$

where

$$\langle f(t,k) \rangle = 0 \quad , \quad \langle f(t,k) f(t',k') \rangle = 2D(2\pi)\delta(t-t')\delta(k+k') \ . \tag{26}$$

We can now solve for v as in the example of a diffusing particle in a parabolic potential:

$$v(t,k) = \mathrm{e}^{-\mu k^2 t} \int_0^t \mathrm{d}s \mathrm{e}^{\mu k^2 s} k f(s,k) \ . \tag{27}$$

What we want to calculate is $\langle v(t,x) v(t,x') \rangle$, which is given by

$$\begin{aligned}
\langle v(t,x) v(t,x') \rangle &= -\frac{1}{(2\pi)^2} \int \mathrm{d}k \mathrm{d}k' \mathrm{e}^{\mathrm{i}(kx+k'x')} \langle v(t,k) v(t,k') \rangle && \text{(28a)} \\
&= -\frac{1}{(2\pi)^2} \int \mathrm{d}k \mathrm{d}k' \Bigg\{ \mathrm{e}^{\mathrm{i}(kx+k'x') - 2\mu k^2 t} \\
&\quad \times \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \mathrm{e}^{\mu k^2 (s+s')} k k' \langle f(s,k) f(s',k') \rangle \Bigg\} && \text{(28b)} \\
&= \int_{-\infty}^{\infty} \frac{Dk^2}{\mu k^2} \mathrm{e}^{\mathrm{i}k(x-x')} && \text{(28c)} \\
&= \frac{D}{\mu} \delta(x-x') \ . && \text{(28d)}
\end{aligned}$$

Comparing this with (22), we can identify the delta function in space with the inverse volume. To doing so, we find that

$$D = \frac{\mu k_B T}{\rho} \ . \tag{29}$$

Extrapolating for $d > 1$, the fluctuating Navier-Stokes equation for thermal, passive fluids is

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{f} \ , \tag{30}$$

where

$$\langle f_i(t,\mathbf{r}) \rangle = 0 \quad , \quad \langle f_i(t,\mathbf{r}) f_j(t',\mathbf{r}') \rangle = \frac{2\mu k_B T}{\rho} \delta_{ij} \delta(t-t') \nabla^2 \delta^d(\mathbf{r}-\mathbf{r}') \ . \tag{31}$$

3 Incompressible active fluids

We will focus exclusively on the so-called "dry" active matter [5, 6], in the sense that there exists a fixed background in the system for the active constituents to exert forces on. Experimentally, the active constituents can be motile cells and the fixed background can be a gel substrate that the cells crawl on. In contrast, wet active matter describes motile organisms in a fluid medium in which organisms move by exchanging momentum with the surrounding fluid, and the resulting fluid flow can in turn affect the motion of the organisms [7, 8].

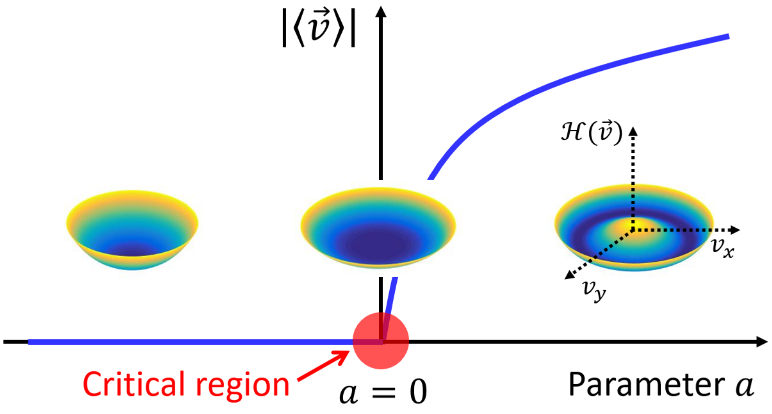

Figure 1: In a generic incompressible active fluid, two distinct phases are possible. At the mean-field level, a disordered phase exists when the parameter a is negative, and an ordered phase (characterised by a non-zero mean speed of the system) emerges when a is positive. The transition between these two phases is continuous, or critical (region depicted in red). The surface plots depicts the "potential energy landscape" at the mean-field level for an active fluid in two dimensions. In the disordered phase, the energy landscape is like a parabolic bowl, while at the transition, the global minimum of the landscape becomes very flat. The landscape transitions further into the shape of a Mexican hat in the ordered phase.

In dry active matter, due to the ability of each active volume element to generate forces against a fixed background, the Galilean invariance no longer applies. Omitting this symmetry, the general EOM of a generic incompressible active fluids is of the form of Eq. (20), which is in fact exactly the incompressible version of the Toner-Tu equation devised to describe the flocking behaviour [2, 3, 9]:

$$\partial_t \mathbf{v} = -\nabla p - \lambda(\mathbf{v} \cdot \nabla)\mathbf{v} + \mu\nabla^2\mathbf{v} + (a - bv^2)\mathbf{v} + \mathbf{f} \tag{32}$$

with blue terms omitted here, and

$$\langle f_i(t, \mathbf{r})\rangle = 0 \quad , \quad \langle f_i(t, \mathbf{r}) f_j(t', \mathbf{r}')\rangle = 2D\delta_{ij}\delta(t - t')\delta^d(\mathbf{r} - \mathbf{r}') \ . \tag{33}$$

Focusing on spatially homogeneous states (so all terms involving ∇ become zero), the simplified EOM can be written as

$$\partial_t \mathbf{v} = -\frac{\delta \mathcal{H}}{\delta \mathbf{v}} \tag{34}$$

where $\mathcal{H}(\mathbf{v}) = -a\mathbf{v}^2/2 + b\mathbf{v}^4/4$. $\mathcal{H}$ can be viewed as a "potential energy" term, whose forms, depending on the parameter a, are depicted in Fig. 1 for a two dimensional system. When a is negative, $\mathcal{H}$ has only one minimum at $\mathbf{v} = 0$, which suggests that the only steady-state solution is the $\mathbf{v} = 0$ homogeneous state. We call this the *disordered* phase. As a increases beyond zero, a continuum of minima emerge and all of these will have a non-zero mean speed. This corresponds to the *ordered* phase, or the *collective motion* phase. The transition between these two phases is continuous and thus constitutes a *critical transition*. From equilibrium statistical mechanics, we know that when spatial heterogeneity and fluctuations are restored, the systems can possess scale-invariant features at criticality [1, 10, 11]. This is also what happens in our active fluid system, as we shall show next.

3.1 Scale invariant properties around the critical point

To understand the emergence of scale-invariant structures at the critical point when the system transitions from the disordered phase to the ordered phase, we will analyse the EOM at the linear level.

3.1.1 Linear theory

To arrive at the linear equation, we tune all the coefficients in the EOM to zero except for the terms below:

$$\partial_t \mathbf{v} = \mu \nabla^2 \mathbf{v} + \mathbf{f} \ , \tag{35}$$

where we have added the Gaussian noise term $\mathbf{f}$. Since we are interested in an incompressible system, we would like the noise to be incompressible as well. In Fourier space, the incompressibility conditions implies $\mathbf{q} \cdot \mathbf{f} = 0$. Now, given any Gaussian noise term $\tilde{\mathbf{f}}$ with statistics

$$\langle \tilde{\mathbf{f}}(\mathbf{r}, t) \rangle = 0 \quad , \quad \langle \tilde{f}_i(\mathbf{r}, t) \tilde{f}_j(\mathbf{r}', t') \rangle = 2D\delta(\mathbf{r} - \mathbf{r}')\delta(t - t') \ , \tag{36}$$

we can use the transverse projection operator $P_{ij}(\mathbf{k}) \equiv \delta_{ij} - k_i k_j / k^2$ to define an incompressible noise term as $f_i = P_{ij} \tilde{f}_j$. Since $\mathbf{k} \cdot \mathbf{f} = k_i P_{ij} \tilde{f}_j = 0$, $\mathbf{f}$ is incompressible as desired. In the Fourier transformed space, $\mathbf{f}$ has the statistics

$$\langle \mathbf{f}(\mathbf{k}, t) \rangle = 0 \tag{37a}$$

$$\begin{aligned} \langle f_i(\mathbf{k}, t) f_j(\mathbf{k}', t') \rangle &= P_{ik}(\mathbf{k}) P_{jh}(\mathbf{k}') (2\pi)^d \langle \tilde{f}_k(\mathbf{k}, t) \tilde{f}_h(\mathbf{k}', t') \rangle \\ &= 2D(2\pi)^d P_{ij}(\mathbf{k}) \delta(\mathbf{k} + \mathbf{k}') \delta(t - t') \ . \end{aligned} \tag{37b}$$

Note that the form of the noise term $\mathbf{f}$ also respects all of the symmetries (symmetries (i)–(iv)) imposed on our system. Furthermore, we no longer need the Lagrange multiplier in the linear EOM (35) as it is intrinsically incompressible.

To investigate the scale invariant properties of our linear model, we now perform the following re-scaling

$$\mathbf{r} \mapsto \mathrm{e}^{\ell} \mathbf{r} \quad , \quad \mathbf{v} \mapsto \mathrm{e}^{\chi \ell} \mathbf{v} \quad , \quad t \mapsto \mathrm{e}^{z\ell} t \ , \tag{38}$$

for some dimensionless number ℓ that describes how the spatial length scale is modified. The field variable $\mathbf{v}$ and time t are also re-scaled, albeit with distinct exponents: the *roughness exponent* χ and the *dynamic exponent* z, respectively. The numerical values of these two exponents are yet to be determined.

Applying the re-scaling to Eq. (35), we find

$$\mathrm{e}^{(\chi - z)\ell} \partial_t \mathbf{v} = \mathrm{e}^{(\chi - 2)\ell} \mu \nabla^2 \mathbf{v} + \mathrm{e}^{-(z+d)\ell/2} \mathbf{f} \ . \tag{39}$$

The prefactor in front of the noise term originates from the form of the noise term (37) and the fact that the delta function scales inversely to its argument, e.g., $\delta(t) \mapsto \mathrm{e}^{-z\ell} \delta(t)$.

Re-writing Eq. (39) as

$$\partial_t \mathbf{v} = \mathrm{e}^{(z-2)\ell} \mu \nabla^2 \mathbf{v} + \mathrm{e}^{(z - 2\chi - d)\ell/2} \mathbf{f} \ , \tag{40}$$

we see that the transformed equation is exactly of the form of the original EOM (39) except that the coefficients $\mu_\ell \equiv \mathrm{e}^{(z-2)\ell} \mu$ and $D_\ell = \mathrm{e}^{(z-2\chi-d)\ell} D$ have acquired a dependency on ℓ. What it means is that if we re-scale the spatial coordinate, then the coefficients in the EOM will generically be modified. We can express the coefficients' dependencies of ℓ in the form of differential equations:

$$\frac{1}{\mu_\ell} \frac{\mathrm{d}\mu_\ell}{\mathrm{d}\ell} = z - 2 \quad , \quad \frac{1}{D_\ell} \frac{\mathrm{d}D_\ell}{\mathrm{d}\ell} = z - 2\chi - d \ . \tag{41}$$

We shall call the above the *flow equations* of the coefficients.

If we now pick z to be 2 and χ to be $(2-d)/2$, then μ_ℓ and D_ℓ remains unchanged as ℓ changes. In other words, given this choice of the exponents, the coefficients in the linear EOM are invariant under re-scaling. The beauty of this invariance is that it enables us to obtain the power-law behaviour of the temporal and spatial correlation functions of the system [12]. For instance, we can relate the equal-time correlation function at different distance because

$$\langle \mathbf{v}(0, t) \cdot \mathbf{v}(r, t) \rangle = \langle \mathbf{v}(0, t) \cdot \mathbf{v}(\mathrm{e}^{\ell}, t) \rangle = \mathrm{e}^{2\chi\ell} \langle \mathbf{v}(0, t) \cdot \mathbf{v}(1, t) \rangle \sim r^{2\chi} \ , \tag{42}$$

where we have picked ℓ such that $\ell = \ln r$, and the second equality follows from the fact that the the re-scaling of $\mathbf{r}$ can be absorbed by re-scaling the field variable $\mathbf{v}$ according to $\mathbf{v} \mapsto e^{\chi\ell}\mathbf{v}$.

What we have seen is that in the linear theory, by suitably re-scaling the field variable and time, the coefficients in the EOM will remain invariant under spatial re-scaling, which leads to a power-law behaviour of the correlation function. Importantly, the power law exponents follow purely from the structure of the equation, and are independent of the actual coefficients in the EOM.

3.1.2 Checking the scaling again

We have seen that the scaling behaviour of the velocity-velocity correlation function can be obtained by suitably choosing the scaling exponents so the EOM remains invariant upon re-scaling. As such, the problem was reduced to solving two independent algebraic equations. The simplicity in finding the solution makes one wonder whether the solutions are trustworthy. Here we will confirm the scaling exponent χ using Fourier transformation for $d = 3$.

Fortunately, we have already done all the hard work because we can write down a similar expression as in (43):

$$\begin{aligned} \langle \mathbf{v}(t,\mathbf{r}) \cdot \mathbf{v}(t,\mathbf{r}') \rangle &= \frac{1}{(2\pi)^2} \int \mathrm{d}^d k \mathrm{d}^d k' \Bigg\{ \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+\mathbf{k}'\cdot\mathbf{r}')-2\mu k^2 t} \\ &\quad \times \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \mathrm{e}^{\mu k^2(s+s')} \langle \mathbf{f}(s,\mathbf{k}) \cdot \mathbf{f}(s',\mathbf{k}') \rangle \Bigg\} , \end{aligned} \tag{43}$$

where the only difference is the lack of $-k^2$ in the integrand due to the fact that the noise term $\mathbf{f}$ here are not momentum conserving. We have also straightforwardly generalised the results for $d > 1$.

Now, using (37), we get

$$\langle \mathbf{f}(s,k) \cdot \mathbf{f}(s',k') \rangle = 2D(2\pi)^d P_{ii}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')\delta(t-t') = 2D(2\pi)^d (d-1)\delta(\mathbf{k}+\mathbf{k}')\delta(t-t') \ . \tag{44}$$

Using the above, we find

$$\langle \mathbf{v}(t,\mathbf{r}) \cdot \mathbf{v}(t',\mathbf{r}') \rangle = \frac{D(d-1)}{2\pi\mu} \int \mathrm{d}^d k \frac{\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}'))}}{k^2} \ . \tag{45}$$

If one applies the Laplacian operator to the correlation function with respect to the spatial coordinate $(\mathbf{r} - \mathbf{r}')$, the $-k^2$ term will drop down from the argument in the exponential function, which neatly cancels out the k^2 term in the denominator. As a result, as in (28c), we know the resulting integral is just a Fourier representation of the delta function. In other words, the correlation function above is the Green's function, which we know in three dimension scales like $|\mathbf{r} - \mathbf{r}'|^{-1}$, thus confirming our result in (42) for $d = 3$.

4 Conclusion

We have seen in these lectures how to formulate of a hydrodynamic theory once the relevant variables and symmetries are identified. Focusing on the linearised theory, we then see some of the techniques useful for identifying the exponents if the theory is scale invariant – i.e., if the EOM can be made invariant upon re-scaling the spatial coordinates. It turns that the scaling relations we found using the linear theory are not exact due to the non-linearities in the EOM. However, these corrections can be taken into account by using dynamical renormalisation group (**RG**) methods [13]. We will go into the details of these methods but it suffices to say that the advent of RG methods in the 1970 has revolutionised how physics is done ever since. There are very good books on this topic if the readers are interested in finding out more about the RG methods [1, 10, 11].

References

[1] Mehran Kardar. Statistical Physics of Fields, 2007.

[2] John Toner and Yuhai Tu. Long-Range Order in a Two-Dimensional Dynamical XY Model: How Birds Fly Together. *Physical Review Letters*, 75(23):4326–4329, 1995.

[3] John Toner and Yuhai Tu. Flocks, herds, and schools: A quantitative theory of flocking. *Physical Review E*, 58(4):4828–4858, 1998.

[4] Michael Stone and Paul M. Goldbart. *Mathematics for physics : a guided tour for graduate students*. Cambridge University Press, 2009.

[5] John Toner, Yuhai Tu, and Sriram Ramaswamy. Hydrodynamics and phases of flocks. *Annals of Physics*, 318(1):170–244, 2005.

[6] M C Marchetti, J F Joanny, S Ramaswamy, T B Liverpool, J Prost, Madan Rao, and R Aditi Simha. Hydrodynamics of soft active matter. *Reviews of Modern Physics*, 85(3):1143–1189, 2013.

[7] Aditi Simha and Sriram Ramaswamy. Hydrodynamic Fluctuations and Instabilities in Ordered Suspensions of Self-Propelled Particles. *Physical Review Letters*, 89(5):58101, 2002.

[8] Yashodhan Hatwalne, Sriram Ramaswamy, Madan Rao, and R Aditi Simha. Rheology of Active-Particle Suspensions. *Physical Review Letters*, 92(11):118101, 2004.

[9] John Toner. Reanalysis of the hydrodynamic theory of fluid, polar-ordered flocks. *Physical Review E*, 86(3):031918, 2012.

[10] John Cardy. *Scaling and Renormalization in Statistical Physics*. Cambridge University Press, 1996.

[11] Nigel Goldenfeld. *Lectures On Phase Transitions And The Renormalization Group (Frontiers in Physics, 85)*. Westview Press, 1992.

[12] A L Barabási and H E Stanley. *Fractal Concepts in Surface Growth*. Cambridge University Press, 1995.

[13] Leiming Chen, John Toner, and Chiu Fan Lee. Critical phenomenon of the order-disorder transition in incompressible active fluids. *New Journal of Physics*, 17:042002, 2015.