Complexity Forum, Warwick University
February 6, 2013

Stochastic dynamics of reaction-diffusion systems: An epidemic model case study

Chiu Fan Lee

Department of Bioengineering

Imperial College London

Imperial College London

Reaction-diffusion systems are ubiquitous

- Extensively used in modelling dynamical processes in physics, chemistry, biology, ecology, etc

- Typical representations: $$\frac{\mathrm{d}\vec{\phi}}{\mathrm{d}t} = \vec{F}(\vec{\phi}) + \mathbf{D} \cdot \nabla^2 \vec{\phi}$$

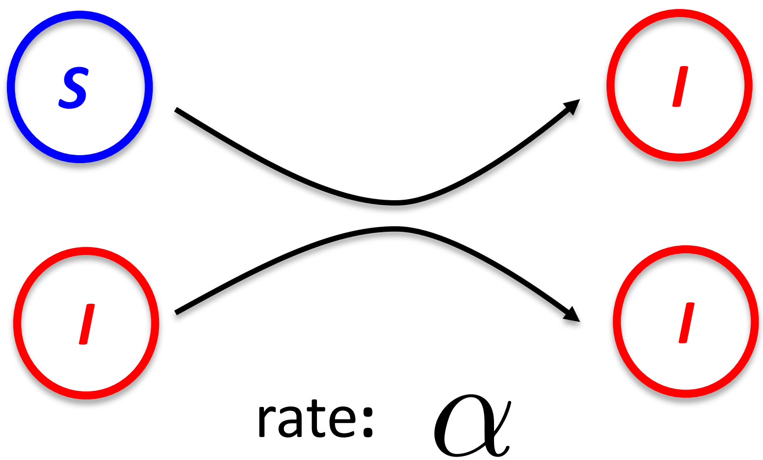
- Problem: Neglect of inevitable intrinsic noises:
  - Reaction & diffusion fluctuations

- Goal: To understand the importance of these fluctuations, and how to incorporate them into the analysis

SIRS epidemic model as a case study

- A infectious disease model with 3 types of individuals
  - S : density of susceptible individuals
  - I : density of infected individuals
  - R : density of recovered individuals

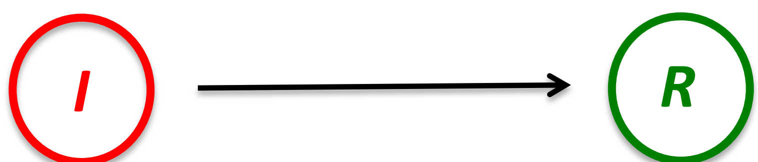
Infection:



SIRS epidemic model as a case study

- A infectious disease model with 3 types of individuals
  - *S* : density of susceptible individuals
  - *I* : density of infected individuals
  - *R* : density of recovered individuals

Recovery:

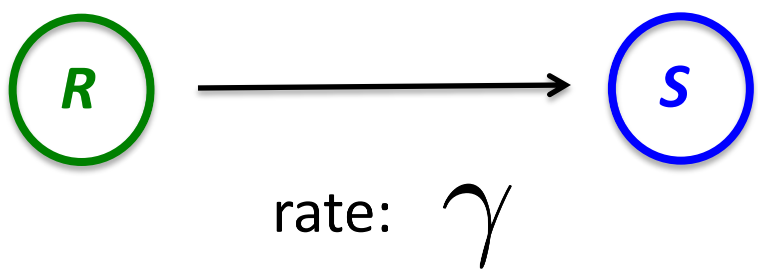


rate: β

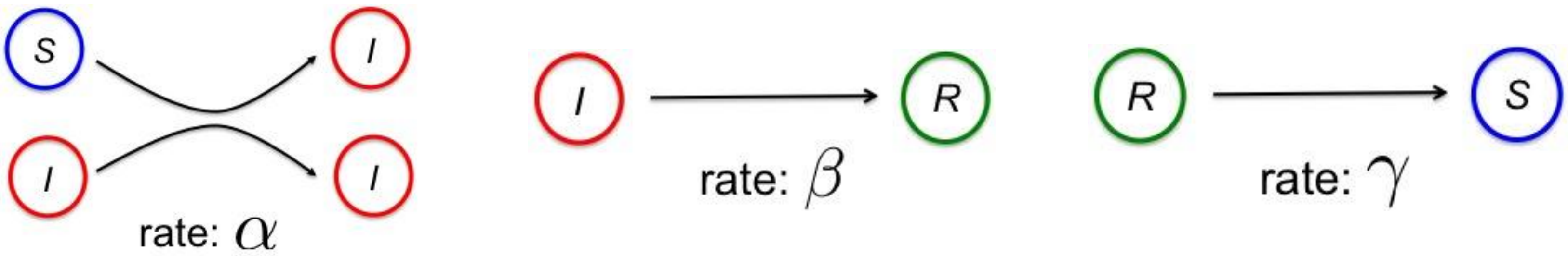
SIRS epidemic model as a case study

- A infectious disease model with 3 types of individuals
  - S : density of susceptible individuals
  - I : density of infected individuals
  - R : density of recovered individuals

Re-infection possible:



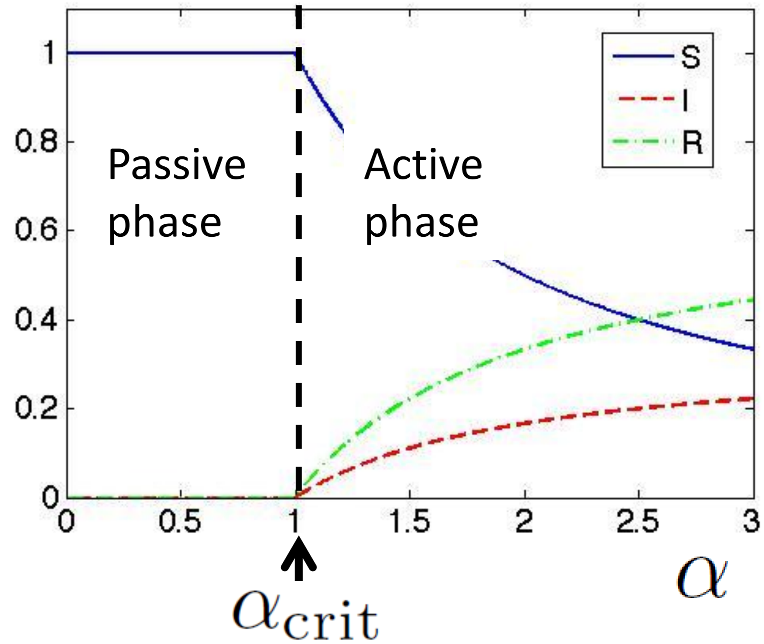
Epidemic evolution



$$
\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= -\alpha S I + \gamma R \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \alpha S I - \beta R \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \beta I - \gamma R
\end{aligned}
$$

Steady-state solution

- 1 conservation law: $$N = S(t) + I(t) + R(t)$$
- 1 stable fixed point: $$[N = 1, \beta = 2, \gamma = 1]$$



How about diffusion?

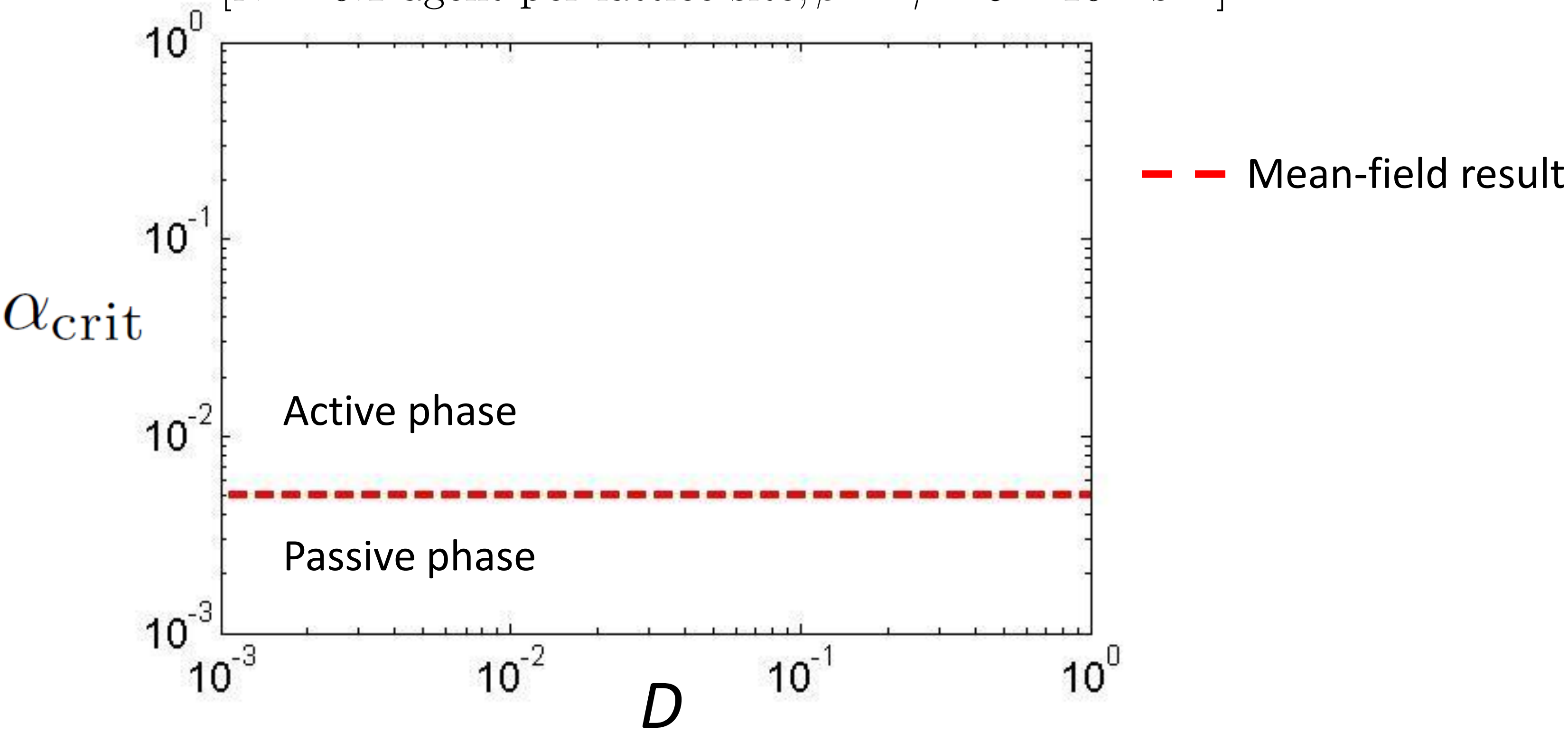
- Typical setup:
$$
\begin{aligned}
\partial_t S &= D\nabla^2 S - \alpha S I + \gamma R \\
\partial_t I &= D\nabla^2 I + \alpha S I - \beta R \\
\partial_t R &= D\nabla^2 R + \beta I - \gamma R
\end{aligned}
$$

- Linear stability from mean-field fixed point at α_{crit} , i.e., consider $S(\mathbf{x}, t) = 1 + \epsilon_S \mathrm{e}^{\sigma t + \mathrm{i}\mathbf{k}\cdot\mathbf{x}}$, etc

- Find: $\sigma = -Dk^2 + (\alpha - \beta)$ ➡ diffusion has no effect on stability

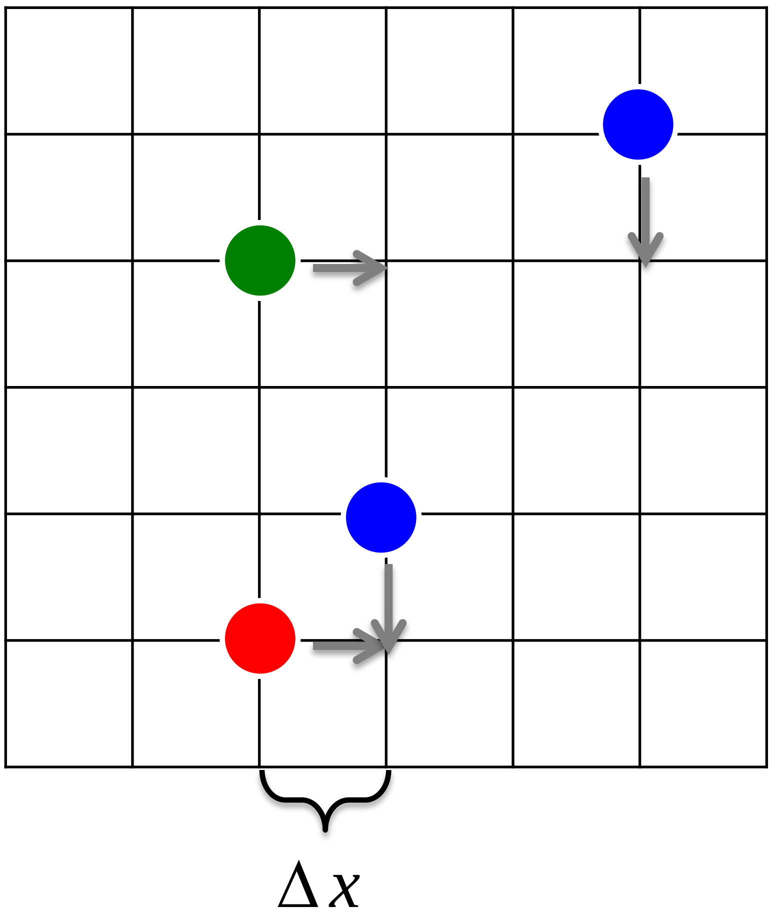
Phase diagram from linear stability analysis

$[N = 0.1 \text{ agent per lattice site}, \beta = \gamma = 5 \times 10^{-4} \text{s}^{-1}]$



Is this true?

What are we actually modelling?
A microscopic picture



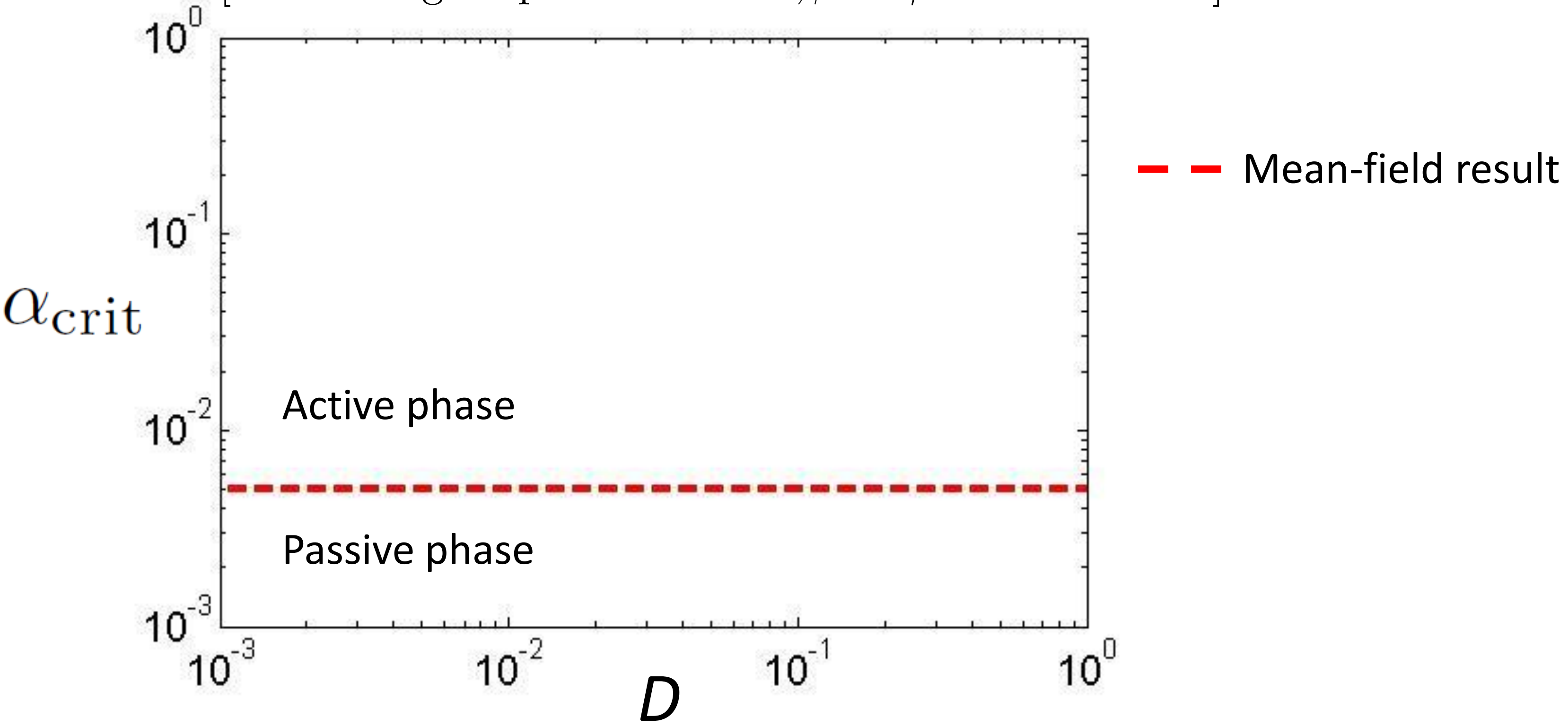
- *S*, *R*, *I* hop to the neighbouring site with rate Γ

$$D = \Gamma \triangle x^2$$

- *S* + *I* ➔ *I* + *I* happens with rate α when *S* and *I* overlap

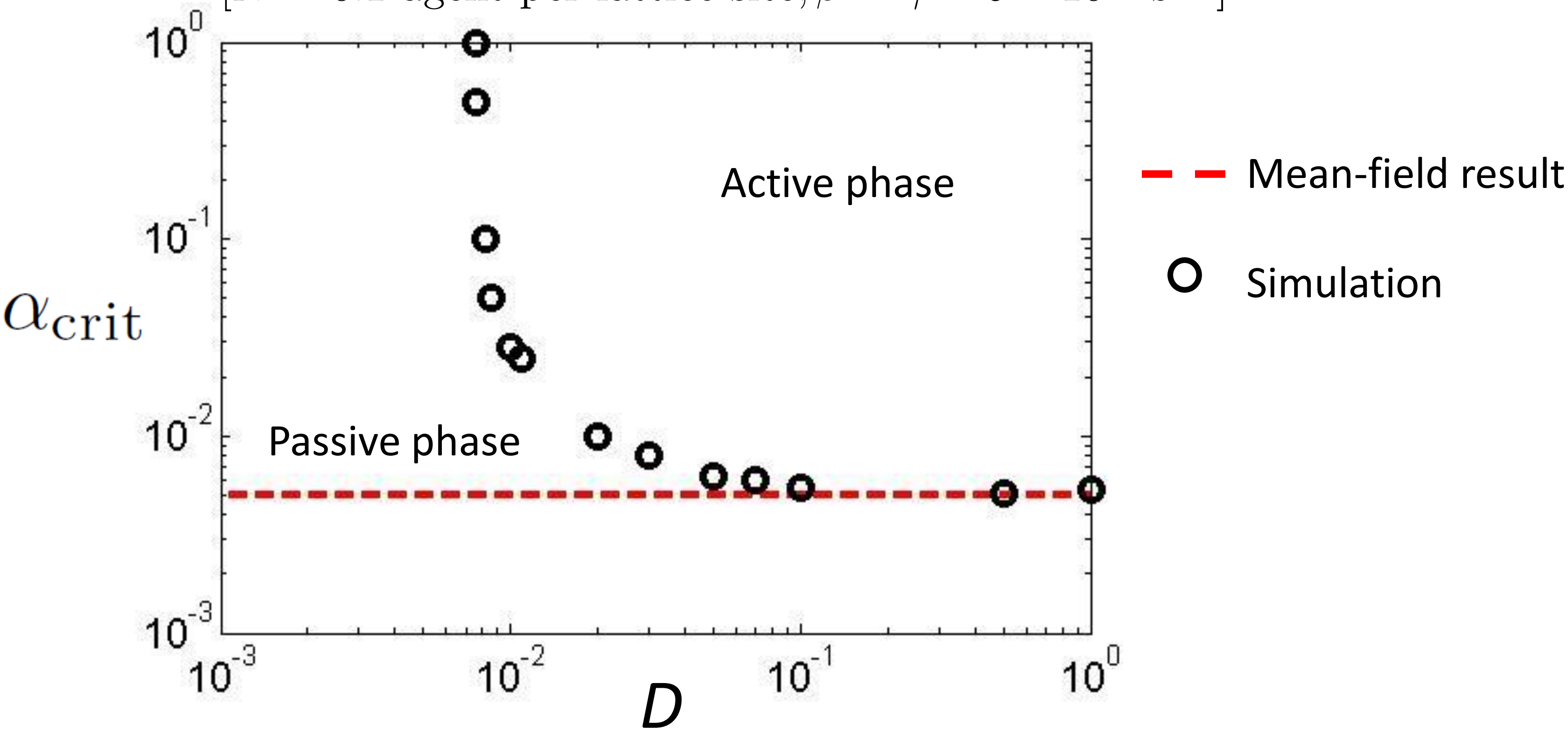
Phase diagram from linear stability analysis

$[N = 0.1 \text{ agent per lattice site}, \beta = \gamma = 5 \times 10^{-4} \text{s}^{-1}]$



Lattice simulation vs. MF result

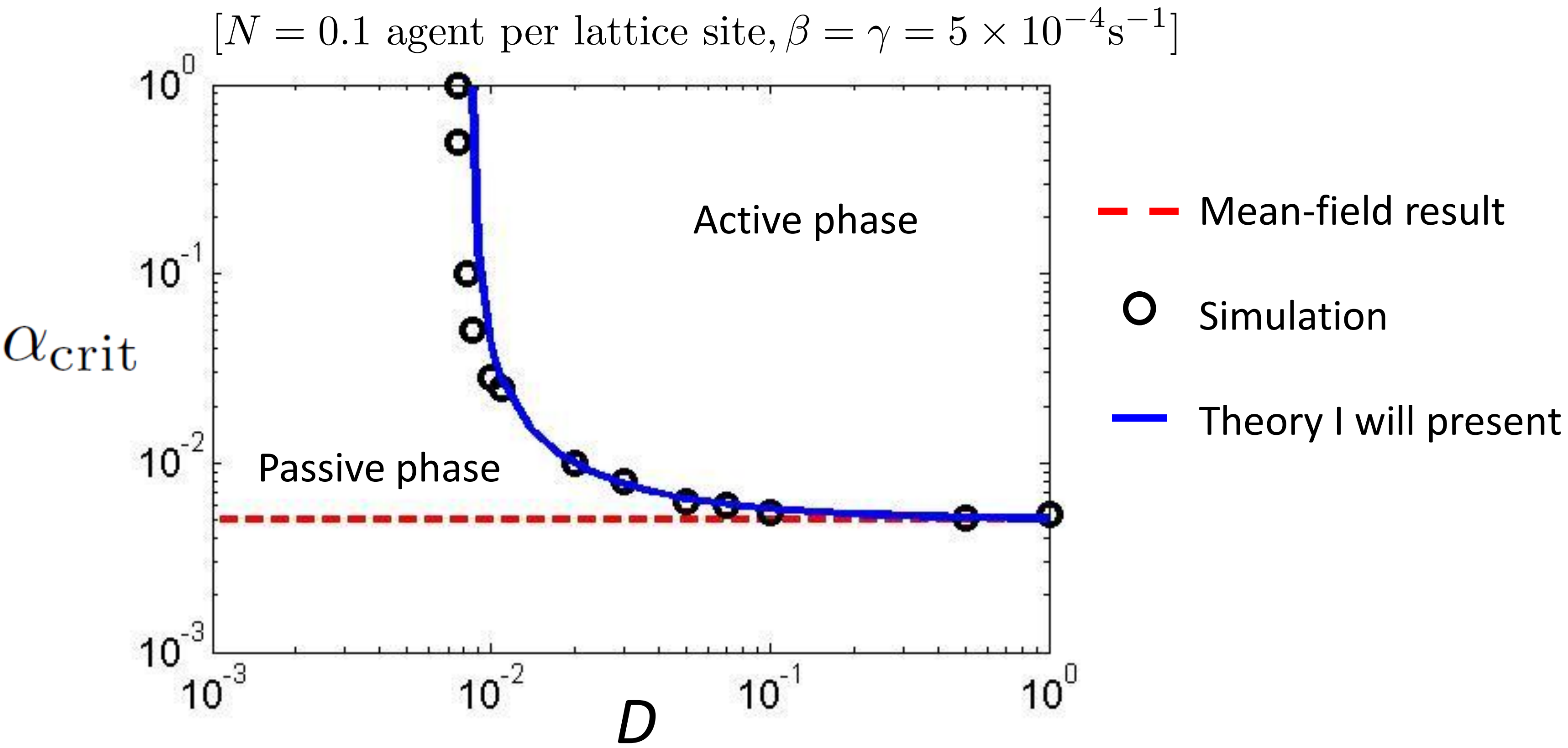
$[N = 0.1 \text{ agent per lattice site}, \beta = \gamma = 5 \times 10^{-4} \text{s}^{-1}]$



What is missing?

Neglect of fluctuations in reaction and diffusion events!

We can do better!



- I will now present a theory that incorporates all the fluctuations into the analysis
- The theory is called the Doi-Peliti formalism and is based on quantum field theory

But first, a bit of history on fluctuation analysis

- Forget about diffusion, and focus on the simple reaction system

$$A + A \xrightarrow{\lambda} \emptyset$$

- Macroscopic equation: $\frac{\mathrm{d}n}{\mathrm{d}t} = -2\lambda n^2$

- To capture fluctuations – Chemical Master Equation (CME):

$$\frac{\mathrm{d}}{\mathrm{d}t} P(n,t) = \lambda \left[(n+2)(n+1)P(n+2,t) - n(n-1)P(n,t) \right]$$

- Unfortunately, the CME is usually intractable

  → need approximations

Kramers-Moyal (KM) expansion

- Start from CME: $\frac{\mathrm{d}}{\mathrm{d}t}P(n,t) = \sum_h [W(n,h)P(h,t) - W(h,n)P(n,t)]$

- Assume transitions only happen in neighbouring states:

$$\begin{aligned}\frac{\partial}{\partial t}P(n,t) &= \int \mathrm{d}h\,[W(n,n-h)P(n-h,t) - W(h,n)P(n,t)] \\ &= \int \mathrm{d}h \sum_{l=1}^{\infty} \frac{(-h)^l}{l!}\frac{\partial^l}{\partial n^l}[\eta_l(n)P(n,t)]\end{aligned}$$

where $\eta_l(n) = \int \mathrm{d}h (n-h)^l W(h,n)$

- For the process: $A + A \xrightarrow{\lambda} \varnothing$, we have

$$\frac{\partial}{\partial t}P(n,t) = \underbrace{2\lambda\frac{\partial}{\partial n}[n^2P(n,t)]}_{\text{Drift}} + \underbrace{2\lambda\frac{\partial^2}{\partial n^2}[n^2P(n,t)]}_{\text{Fluctuations}}$$

Drift

Fluctuations

van Kampen approximation

- Also known as system size expansion, Ω expansion, or linear noise approximation
- Define new variables: $n(t) = \Omega\phi(t) + \sqrt{\Omega}x(t)$
- Substitute back into KM expansion to get two equations:
  - One for the intensive variable and one for the fluctuations

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -2\lambda\phi^2$$

$$\frac{\partial}{\partial t}P(x,t) = 4\lambda\phi(t)\frac{\partial}{\partial x}[xP(x,t)] + \lambda\phi(t)^2\frac{\partial^2}{\partial x^2}P(x,t)$$

Problems

- These approximations fail if the system size Ω (i.e., volume, total particle numbers, etc) is small

- Population can become negative due to fluctuations – boundary conditions not captured properly

- **Solutions:** keep track of the boundary condition and the discrete nature of problem seriously by using raising and lowering operators

Operator formalism

- Start with the empty state, the state with no particles: $|0\rangle$
- Define raising operator: $a^\dagger|n\rangle = |n+1\rangle$
  and lowering operator: $a|n\rangle = n|n-1\rangle$
- In vector and matrix form:

$$|0\rangle = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad a^\dagger = \begin{pmatrix} \ddots & & & \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & 0 & 0 & 0 \end{pmatrix} \qquad a = \begin{pmatrix} \ddots & & & \\ & 0 & 0 & 0 \\ & 2 & 0 & 0 \\ & 0 & 1 & 0 \end{pmatrix}$$

- In particular: $a^\dagger a|n\rangle = n|n\rangle \qquad [a, a^\dagger] \equiv aa^\dagger - a^\dagger a = 1$

Operator representation of CME

- Back to $A + A \xrightarrow{\lambda} \emptyset$, with CME:

$$\frac{\mathrm{d}}{\mathrm{d}t}P(n,t) = \lambda\left[(n+2)(n+1)P(n+2,t) - n(n-1)P(n,t)\right]$$

- To use operators, first define: $|\psi(t)\rangle \equiv \sum_n P(n,t)|n\rangle$

- Then

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \lambda(1 - a^{\dagger 2})a^2 \sum_n P(n,t)|n\rangle$$

$$= -\hat{H}|\psi(t)\rangle$$

← Schrödinger equation

- Formal solution: $|\psi(t)\rangle = \exp(-\hat{H}t)|\psi(0)\rangle$

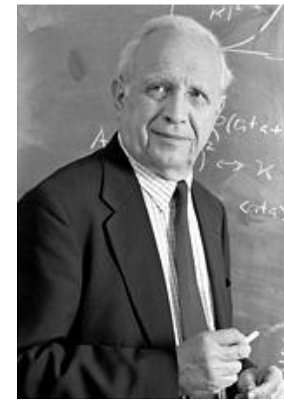
Erwin Schrödinger, Nobel Prize ‘33

Getting rid of non-commutative operators

- Introduce coherent states:

$$|\phi\rangle = \exp\left(-\frac{1}{2}|\phi|^2 + \phi a^\dagger\right)|0\rangle$$

where φ is a complex number



Roy Glauber
Nobel Prize '05

- Slice up the temporal evolution:

$$\mathrm{e}^{-\hat{H}t} = \int \ldots |\phi_{t+\triangle t}\rangle\langle\phi_{t+\triangle t}|\mathrm{e}^{-\hat{H}\triangle t}|\phi_t\rangle\langle\phi_t|\mathrm{e}^{-\hat{H}\triangle t}|\phi_{t-\triangle t}\rangle\langle\phi_{t\triangle t}|\ldots$$

Path integral method

Richard Feynman
Nobel Prize '65

Field theory—it's all about integration

- Do a lot of integration to compute average density

$$\bar{n}(t) = \int \mathcal{D}\bar{\phi}\mathcal{D}\phi\phi(t)\mathrm{e}^{-S}$$

where

$$S = \int \mathrm{d}^d x \mathrm{d}t \left[\bar{\phi}(\partial_t - D\nabla^2)\phi + 2\lambda\bar{\phi}\phi^2 + \lambda\bar{\phi}^2\phi^2 - n_0\bar{\phi}\delta(t)\right]$$

$$\mathcal{D}\bar{\phi}\mathcal{D}\phi = \Pi_k \left(\frac{\mathrm{d}(\mathrm{Im}\phi_{t_k})\mathrm{d}(\mathrm{Re}\phi_{t_k})}{\pi}\right)$$

Doi-Peliti vs. Kramers-Moyal

- In terms of Langevin's equations:

Kramers-Moyal: $$\mathrm{d}\bar{n} = -2\lambda\bar{n}^2\mathrm{d}t + \sqrt{\lambda}\bar{n}\mathrm{d}w$$ ✗

Doi-Peliti: $$\mathrm{d}\bar{n} = -2\lambda\bar{n}^2\mathrm{d}t + \mathrm{i}\sqrt{\lambda}\bar{n}\mathrm{d}w$$ ✓

Nothing's perfect …

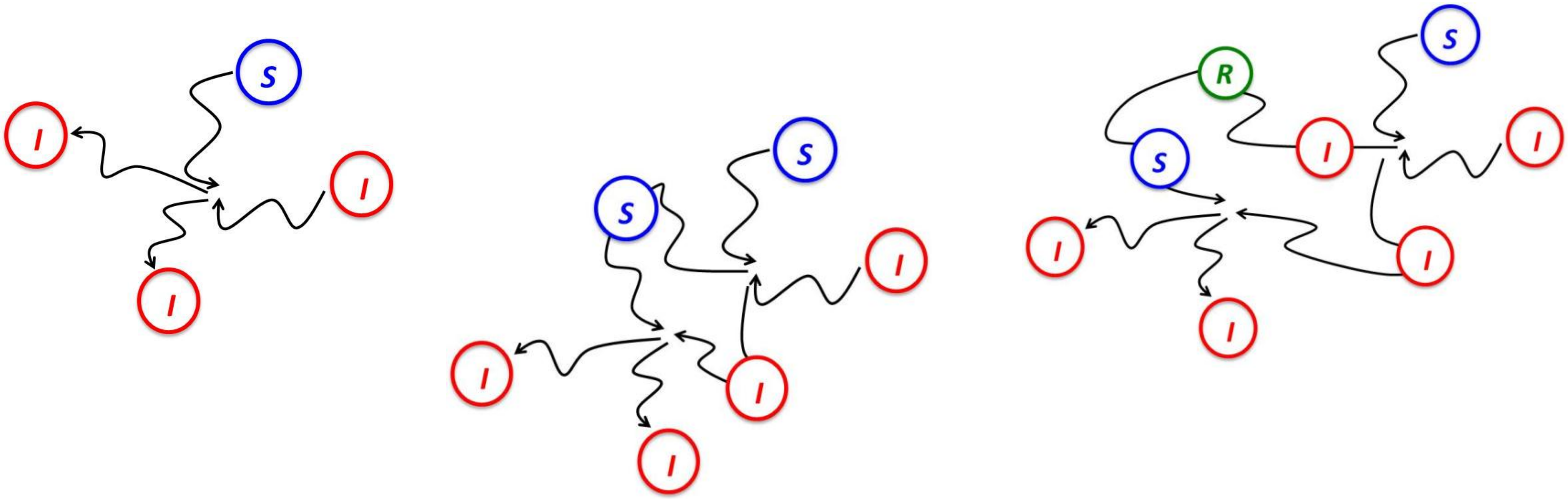
- Problem: we can't really do the integration!

$$\bar{n}(t) = \int \mathcal{D}\bar{\phi}\mathcal{D}\phi\,\phi(t)\mathrm{e}^{-S}$$

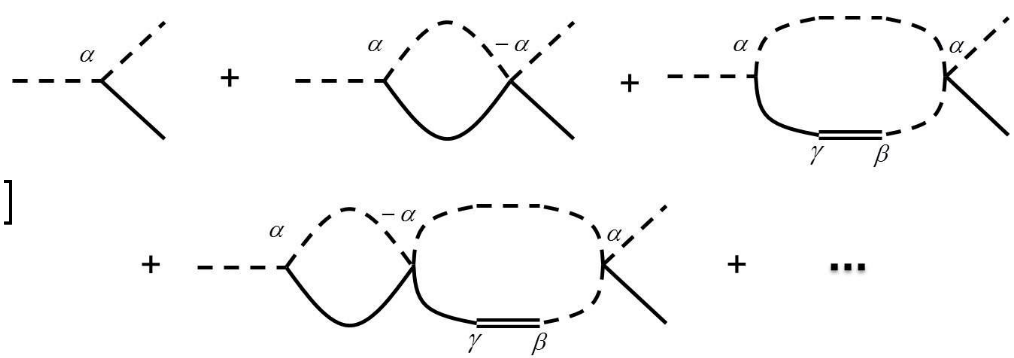
- But: Doi-Peliti formalism enables us to do the integration partially based on physical pictures

So what did I calculate?

- Specifically, I considered the following interactions in the calculation:

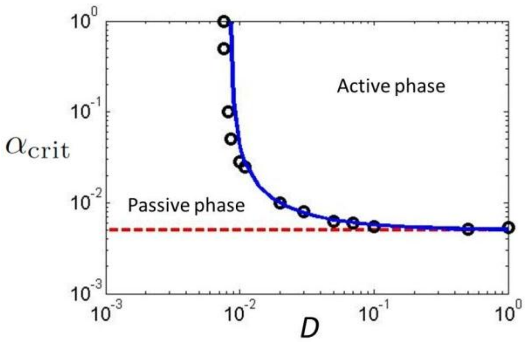


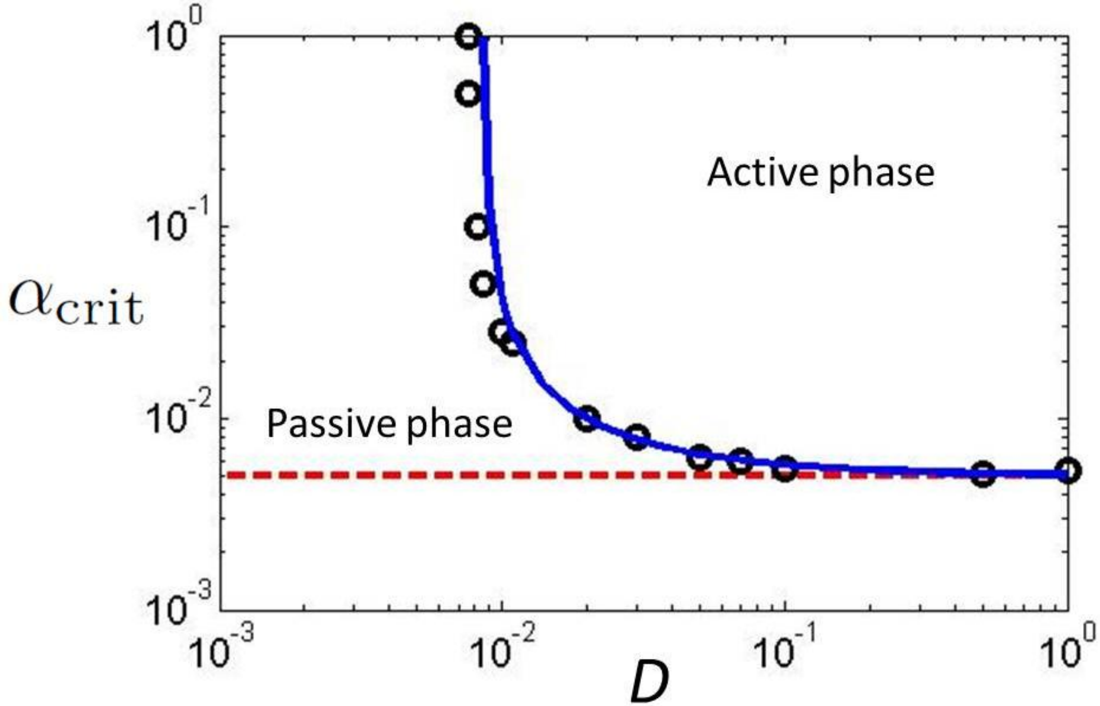
[For the experts, I sum over all the sequential one-loop diagrams]



Agreements with simulation results

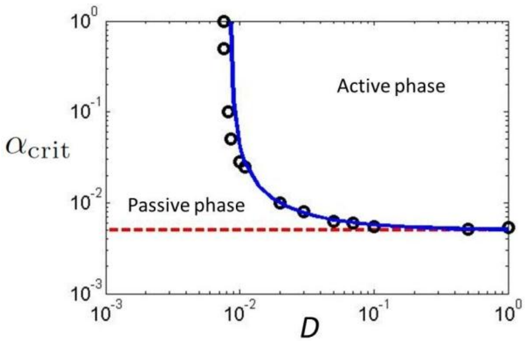
$[N = 0.1 \text{ agent per lattice site}, \beta = \gamma = 5 \times 10^{-4} \mathrm{s}^{-1}]$

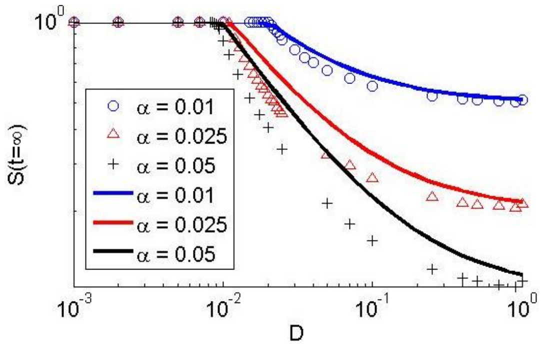


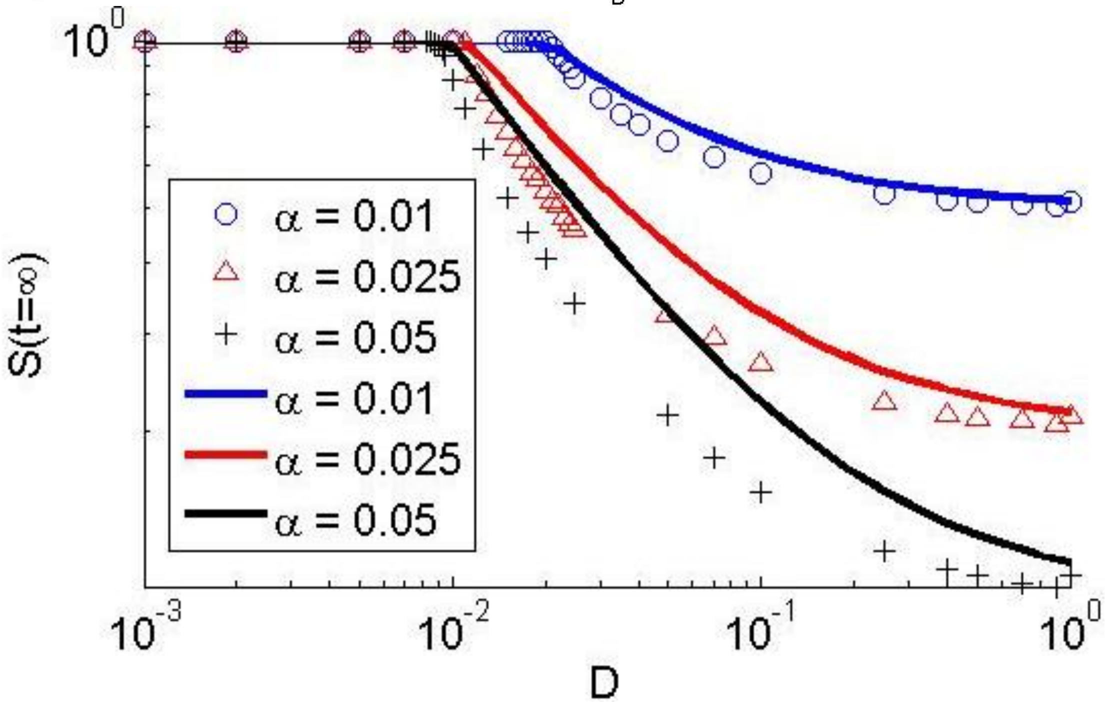


Agreements with simulation results

$$[N = 0.1 \text{ agent per lattice site}, \beta = \gamma = 5 \times 10^{-4} \text{s}^{-1}]$$

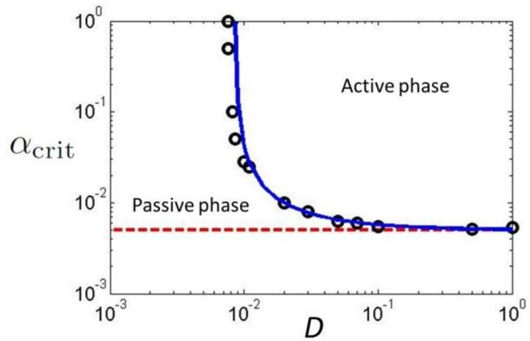


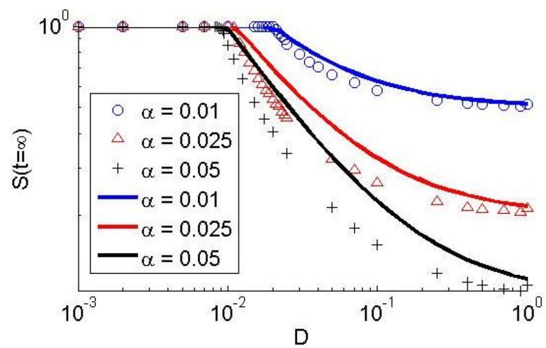


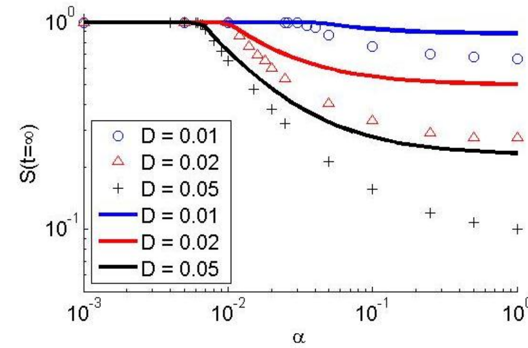


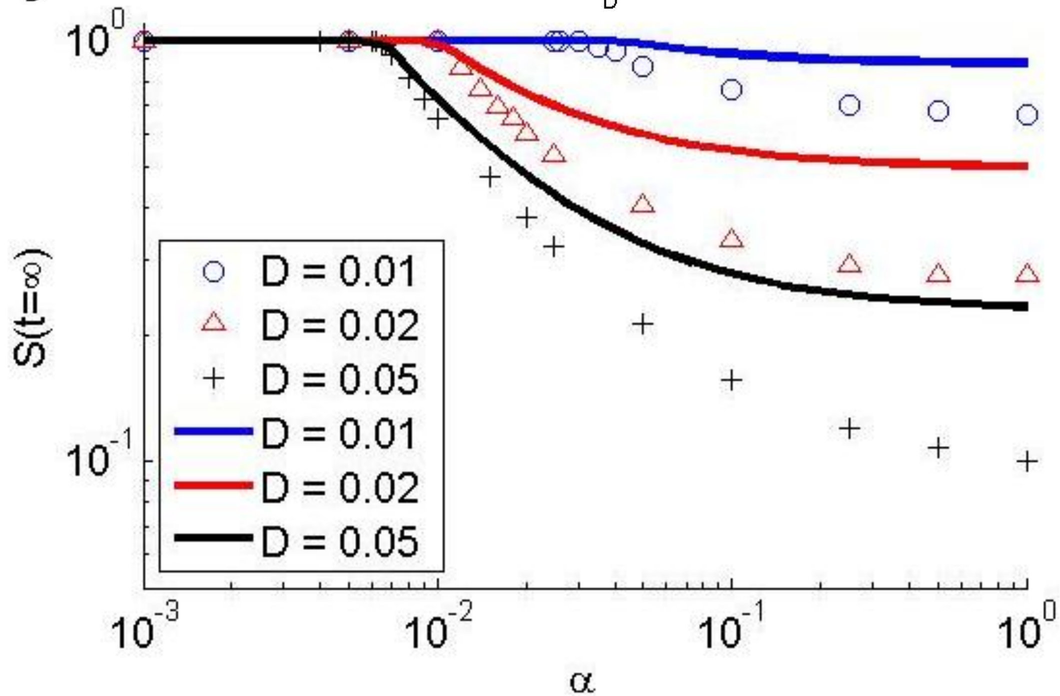
Agreements with simulation results

$$[N = 0.1 \text{ agent per lattice site}, \beta = \gamma = 5 \times 10^{-4} \text{s}^{-1}]$$









Conclusion

- Mobility and fluctuations have critical effects on reaction-diffusion systems
- Doi-Peliti field-theoretical formalism is a method to analyse these effects
- Need further method development to provide a more comprehensive treatment of the problem

Thanks

Fernando Peruani

Laboratoire J.A. Dieudonné, Université de Nice—Sophia Antipolis

mpipks

Max Planck Institute for the
Physics of Complex Systems
Dresden, Germany

References

- Today's talk:

  F. Peruani and C.F. Lee. Fluctuations and the role of collision duration in reaction-diffusion systems. *In preparation.*

- References on the Doi-Peliti method:

  –John Cardy. Reaction-Diffusion Processes. *Available on his homepage.*

  –U.C. Täuber, M. Howard and B.P. Vollmayr-Lee (2005) Applications of field-theoretic renormalization group methods to reaction–diffusion problems. *Journal of Physics A* **38**, R79.

Thank you!