Hydrodynamic Approaches to Active Matter: Successes and Future Challenges
Max Planck Institute for the Physics of Complex Systems
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The Fate of Collective Order in the Active Ising Model

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Bioengineering Department

IMPERIAL



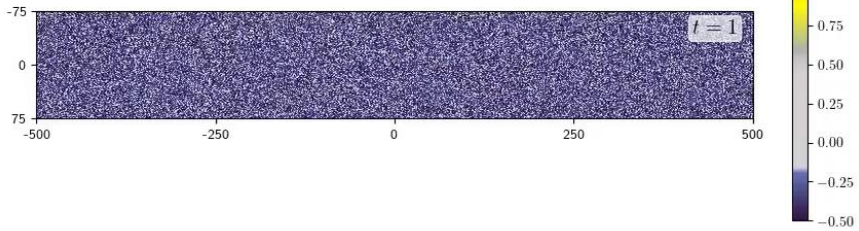
Key question

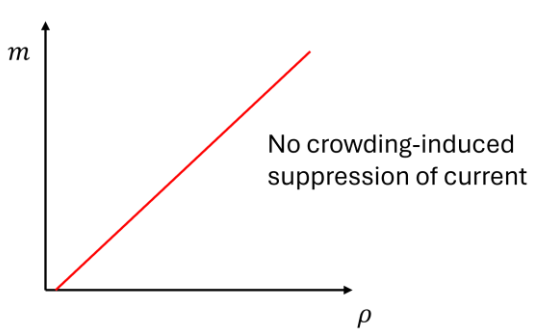
- The Ising model has an ordered phase (for $D \geq 2$)
- Does the ordered phase exist in the “active” Ising model?

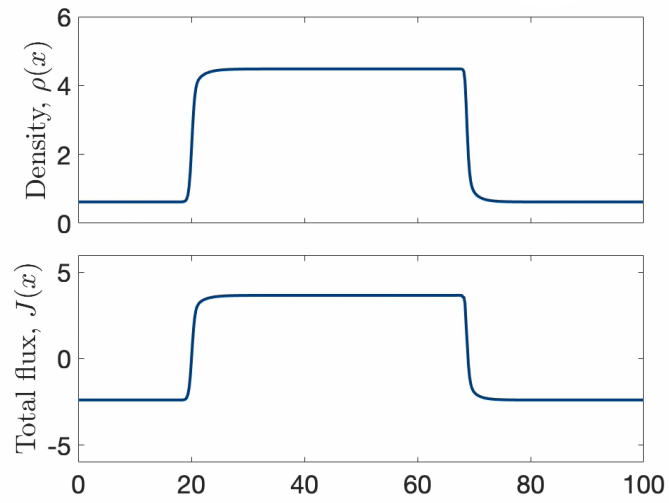
The argument in a nutshell

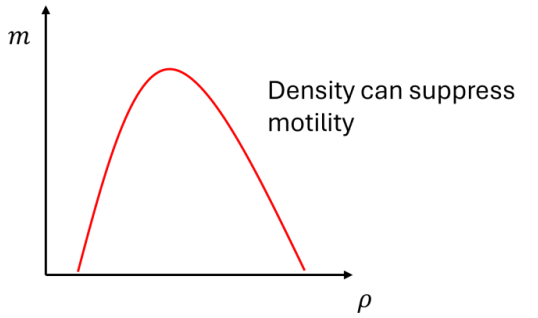
- Benvegnen et al. (2023) argues that the ordered phase of an active Ising model (AIM) is generically unstable because
  - fluctuation can nucleate a dense drop pointing in the opposite direction of the ordered bath
  - This counter propagating drop grows ballistically
- Counterpropagating droplets need not grow: a counterexample already exists in another version of AIM
- What is “the” AIM? Symmetry consideration → Toner-Tu model with an easy axis
- The crucial distinction: density-dependent motility
- Conclusion: Activity does not generically destroy collective order

B. Benvegnen, et al (2023) Metastability of Discrete-Symmetry Flocks. Physical Review Letters 131, 218301







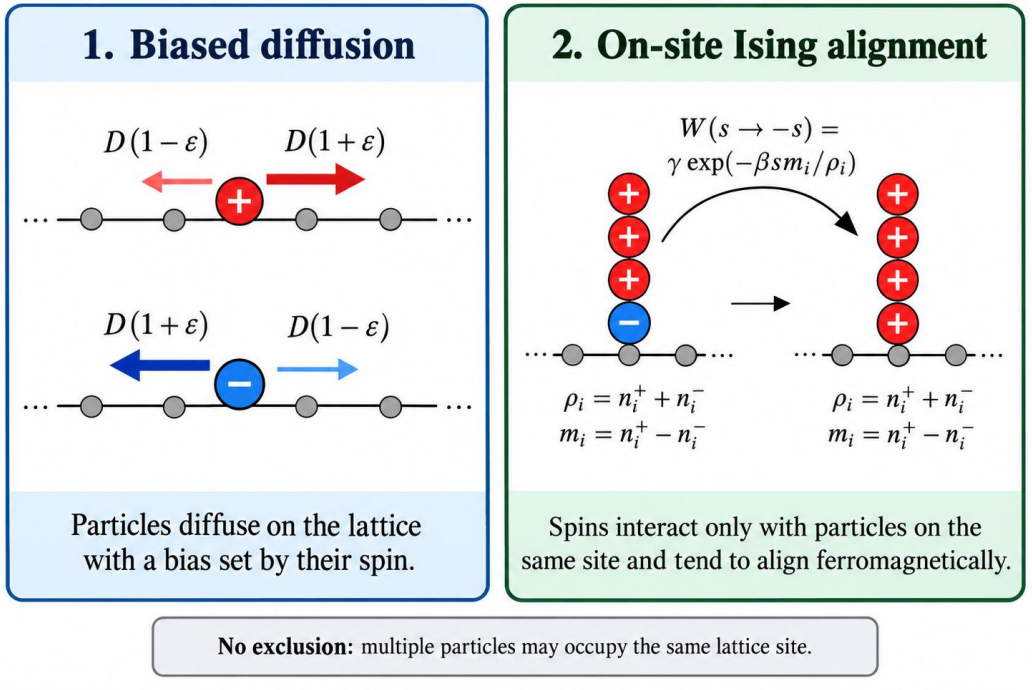


T. Bertrand and C. F. Lee (2022) Diversity of Phase Transitions and Phase Separations in Active Fluids. Physical Review Research 4, L022046

An Active Ising model (AIM)

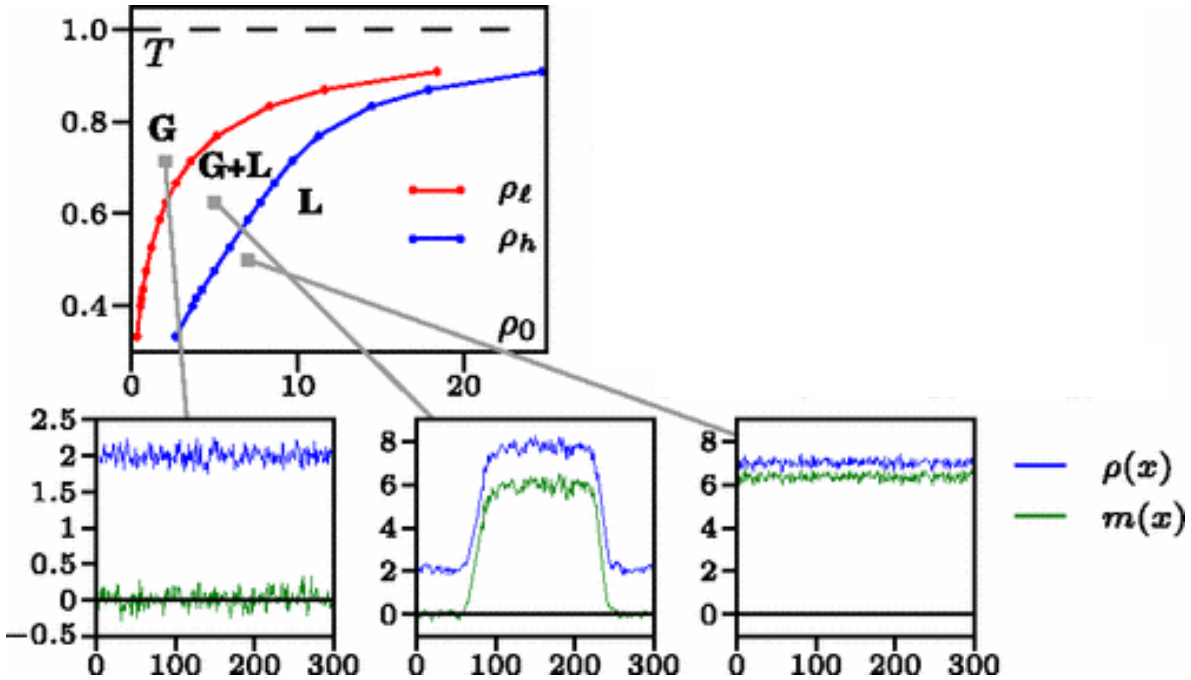
- AIM = biased diffusion + on-site mean-field Ising alignment

[A. P. Solon and J. Tailleur (2013) Revisiting the Flocking Transition Using Active Spins. Physical Review Letters 111, 078101]



Schematics generate by ChatGPT

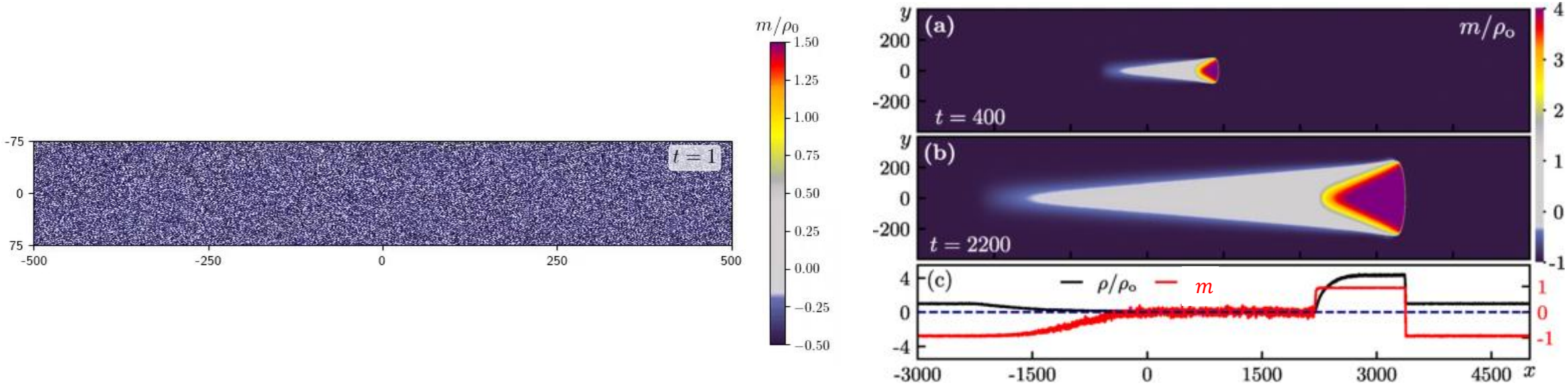
Phase diagram in 2D



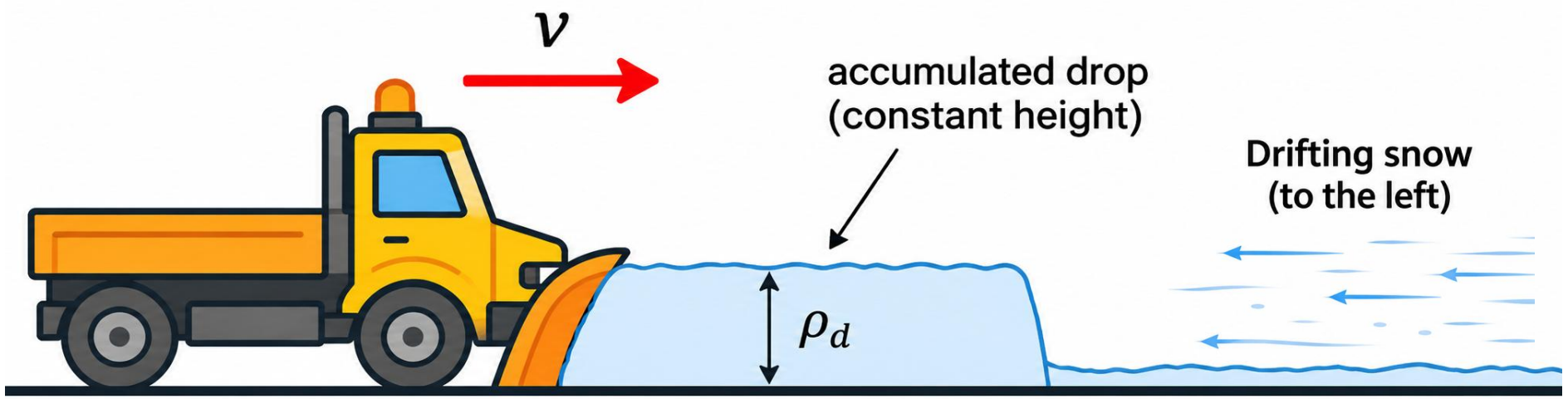
Ordered phase of AIM is not stable

[B. Benvegnen, O. Granek, S. Ro, R. Yaacoby, H. Chaté, Y. Kafri, D. Mukamel, A. Solon, and J. Tailleur (2023) Metastability of Discrete-Symmetry Flocks. Physical Review Letters 131, 218301]

- Their proposed mechanism:
  1. Rare fluctuations nucleate counterpropagating dense droplets
  2. These droplets grow ballistically, destabilising the ordered phase in D ≥ 2



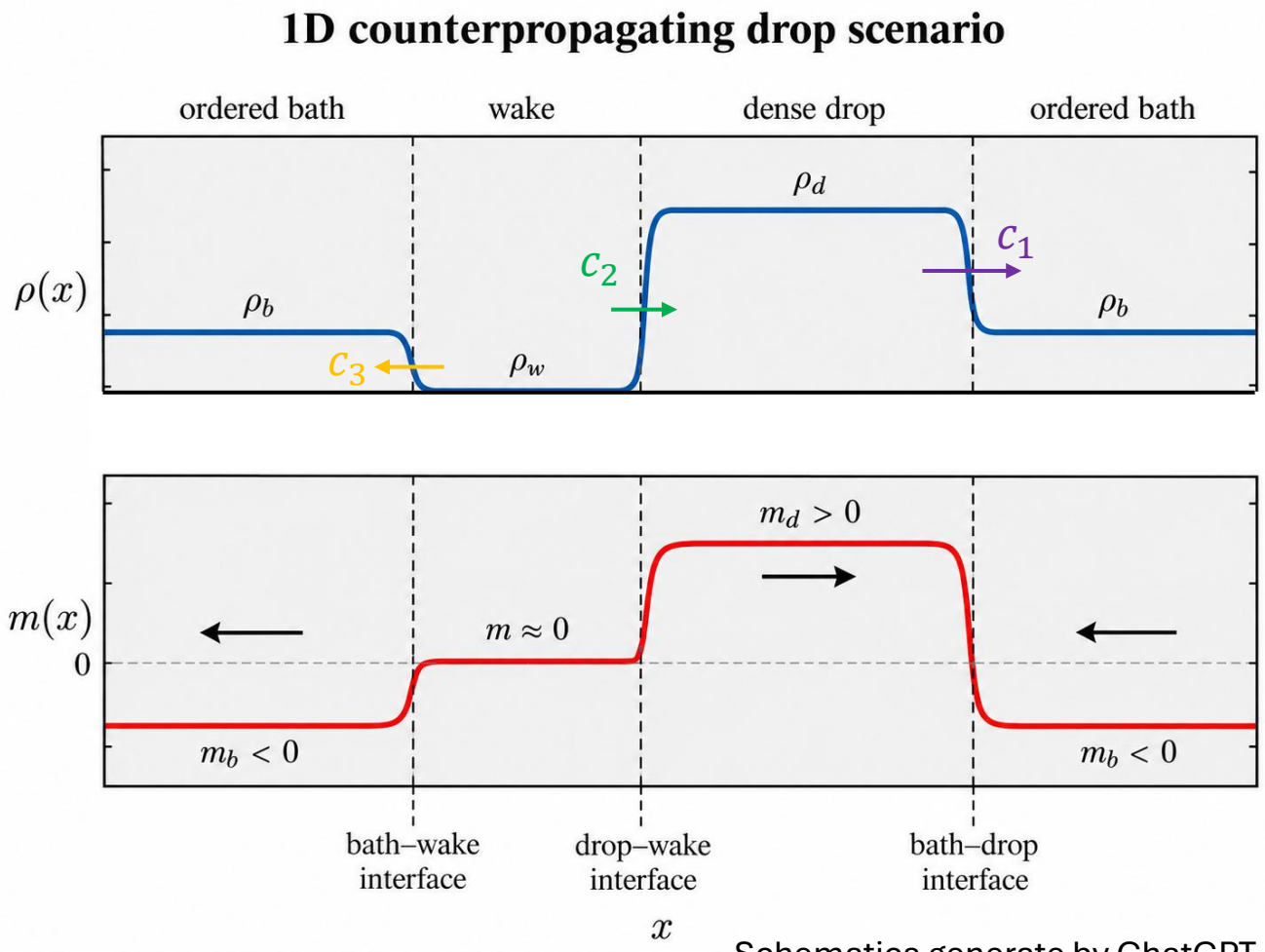
Snowplough mechanism



Schematics generate by ChatGPT

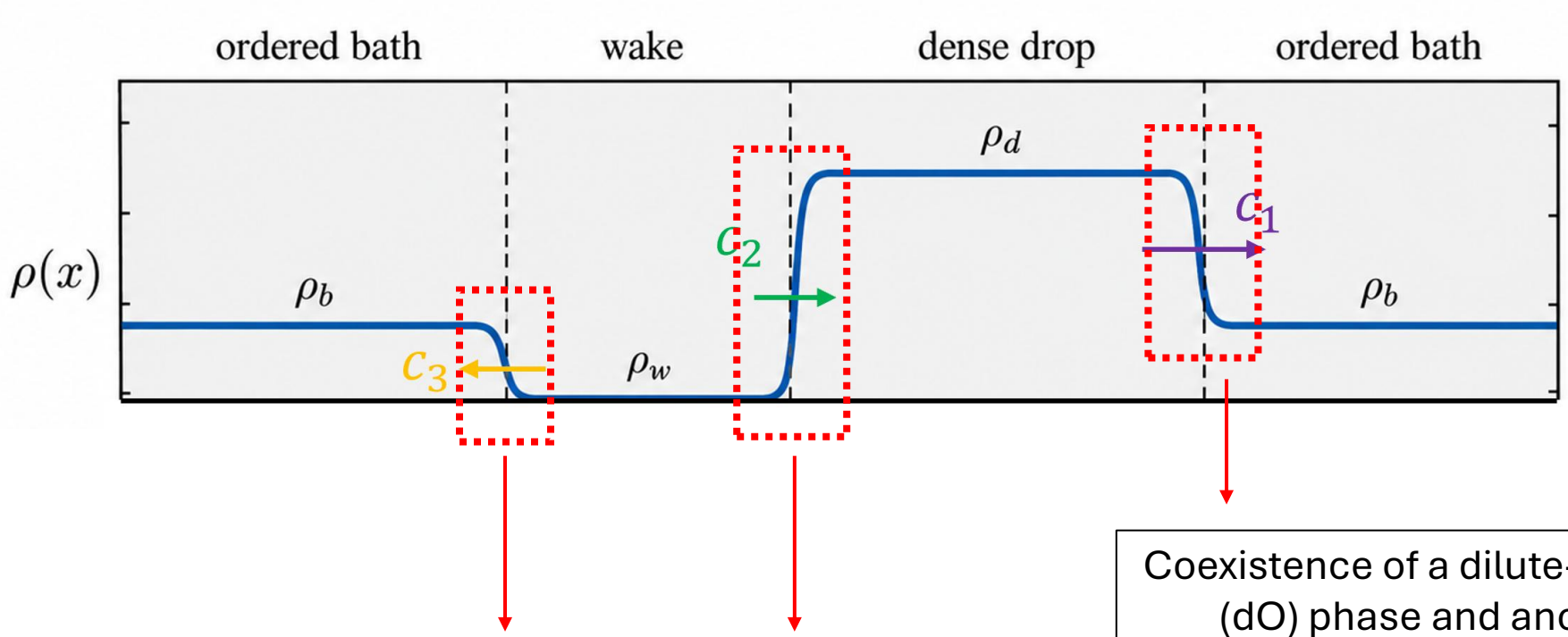
Characteristics

- ρ_d is constant
- $\rho_w \approx 0$
- Interfacial speeds c_1, c_2, c_3 are constant
- $c_1 > c_2$ → ballistic growth



Schematics generate by ChatGPT

Three interfaces



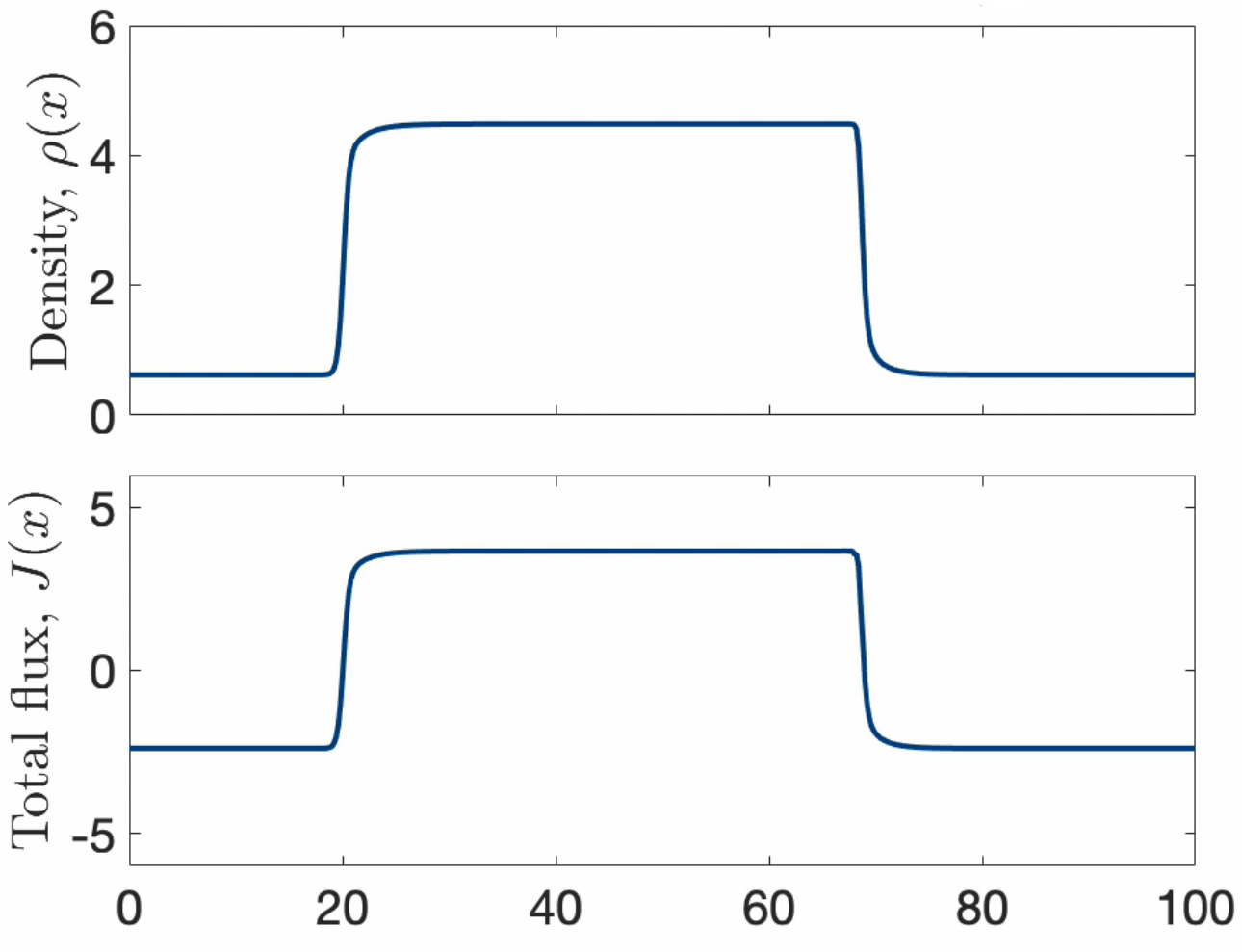
Coexistence of a condensed-Ordered (cO) phase and a dilute Disordered (dD) phase
≈ Travelling band (known)

Coexistence of a dilute-Ordered (dO) phase and another condensed Disordered (cD) phase
≈ We've also seen this before

Coexistence of counter-moving phases

[T. Bertrand and C. F. Lee (2022) Diversity of Phase Transitions and Phase Separations in Active Fluids. Physical Review Research 4, L022046]

- A counter-moving dense drop in an ordered bath
- But the drop is NOT expanding





Let’s dive deeper into phase separation

Diversity of phase separations in polar active fluids

- Phase separation of one-component system → coexistence of a dilute phase (d) and a condensed phase (c)
- In polar active fluids, each phase can be ordered (O) or disordered (D)
- Question: How many distinct coexistences can we have?

dilute	condensed
D	D
D	O
O	D
O	O

← Motility-induced phase separation
[Cates & Tailleur (2015) Ann Rev Condens Matter Phys]

← Banding regime
[Grégoire and H. Chaté (2004) PRL; Chaté et al. (2008) PRE]

← Reverse banding regime
[Nesbitt, Pruessner & Lee (2021) NJP; see also Schnyder et al. (2017) Sci. Rep.; Geyer et al. (2019) PRX]

← Co-moving and counter-moving phases
[Bertrand & Lee (2022) PRR]

dilute	condensed
D	D
D	O
O	D
O	O

Can one hydrodynamic framework capture all four coexistences?

But, what is “the” Active Ising Model?

- **Equilibrium Ising model:** A single scalar order parameter ϕ with $\mathbb{Z}_2$ symmetry: $\phi \to -\phi$
- **Active Ising model:** Spins move preferentially along their orientation (easy axis)
- $\mathbb{Z}_2$ symmetry becomes: $x \to -x, \qquad \phi \to -\phi$
- **Additionally, particle-number conservation:** spins can flip, but particles are neither created nor destroyed
- The general hydrodynamic theory is Toner–Tu with an easy axis

$$\partial_t \rho = -\partial_x m + \eta_x \partial_x^2 \rho + \eta_\perp \nabla_\perp^2 \rho,$$

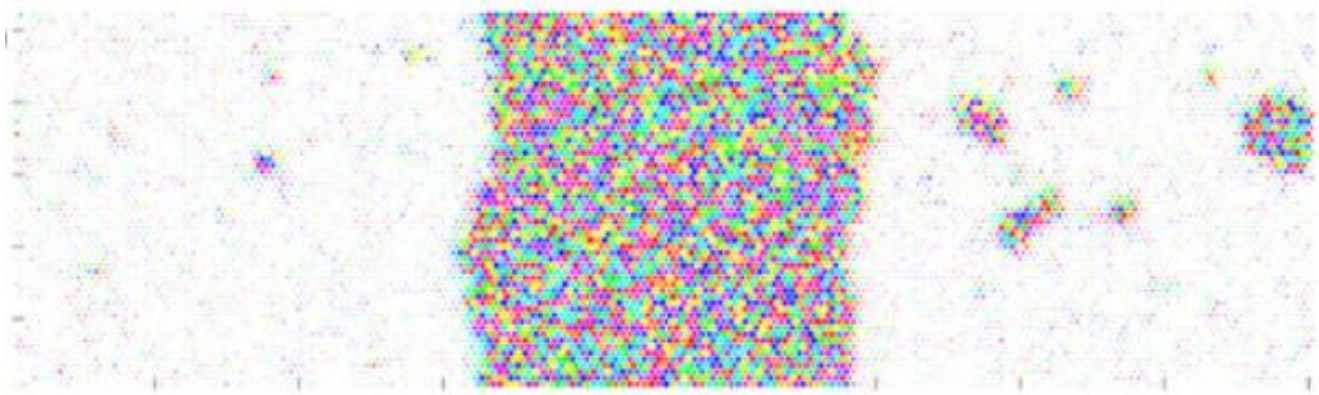
$$\partial_t m + \lambda m \partial_x m = F(\rho, m) - \kappa(\rho)\partial_x \rho + \mu_x \partial_x^2 m + \mu_\perp \nabla_\perp^2 m$$

dD-cD - MIPS

$$\partial_t \rho = -\partial_x m + \eta_x \partial_x^2 \rho + \eta_\perp \nabla_\perp^2 \rho,$$
$$\partial_t m + \lambda m \partial_x m = F(\rho, m) - \kappa(\rho)\partial_x \rho + \mu_x \partial_x^2 m + \mu_\perp \nabla_\perp^2 m$$

- m is damped → $F(\rho, m) \sim -m$
- m ~ $-\kappa(\rho)\partial_x \rho$ → phase separation occurs if $\kappa(\rho) < 0$ for some range of ρ,
- e.g.,

$$\kappa(\rho) = 140 - 145\rho + 30\rho^2$$



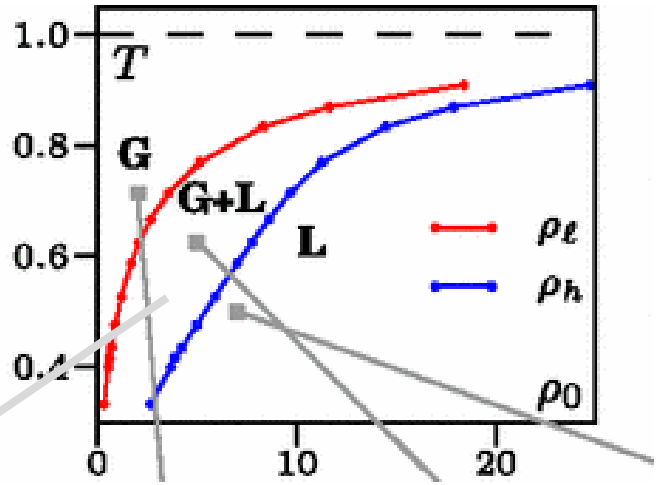


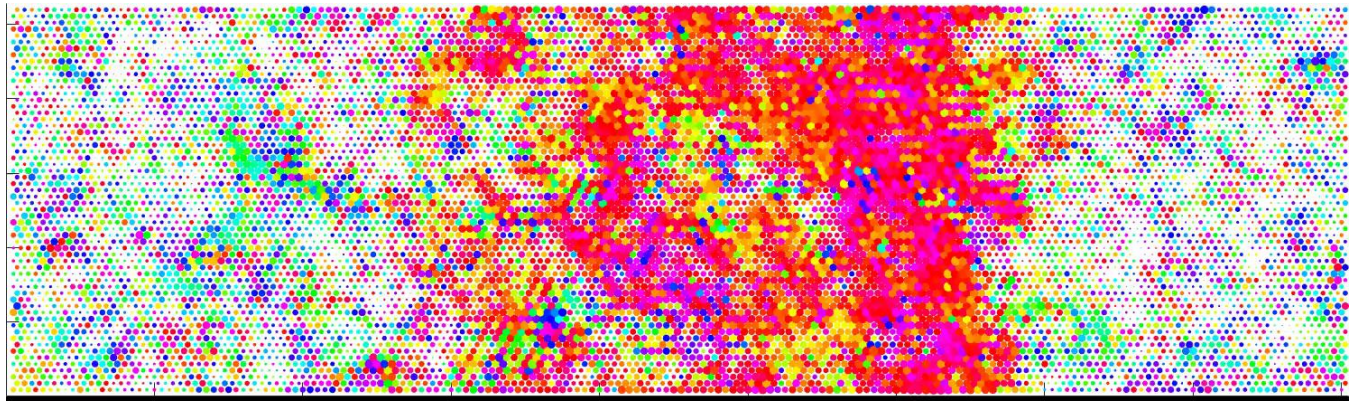
Colour wheel used to determine direction of velocity

dD-cO - Banding

$$\partial_t \rho = -\partial_x m + \eta_x \partial_x^2 \rho + \eta_\perp \nabla_\perp^2 \rho,$$
$$\partial_t m + \lambda m \partial_x m = F(\rho, m) - \kappa(\rho)\partial_x \rho + \mu_x \partial_x^2 m + \mu_\perp \nabla_\perp^2 m$$

- $F(\rho, m) = \alpha m - \beta m^3, \quad \alpha > 0$ and $\alpha \sim 0$







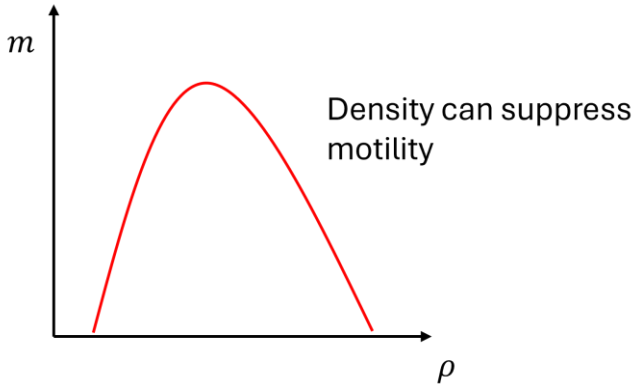
Colour wheel used to determine direction of velocity

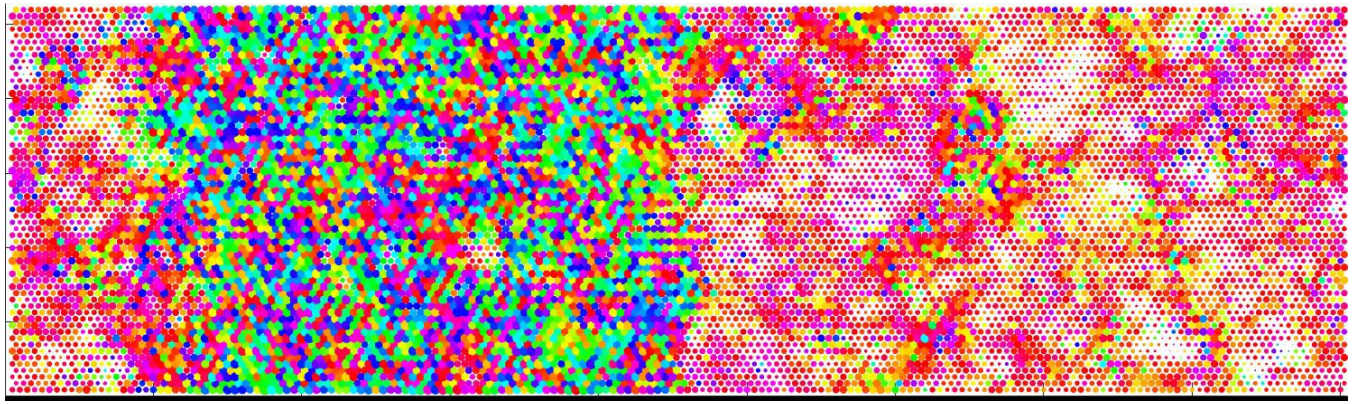
dO-cD – Reverse banding

$$\partial_t \rho = -\partial_x m + \eta_x \partial_x^2 \rho + \eta_\perp \nabla_\perp^2 \rho,$$
$$\partial_t m + \lambda m \partial_x m = F(\rho, m) - \kappa(\rho)\partial_x \rho + \mu_x \partial_x^2 m + \mu_\perp \nabla_\perp^2 m$$

- Motility is inhibited at high density, e.g.,

$$F(\rho, m) = \left(-A + 18\rho - \frac{10}{3}\rho^2\right) m - \frac{1}{2} m^3$$

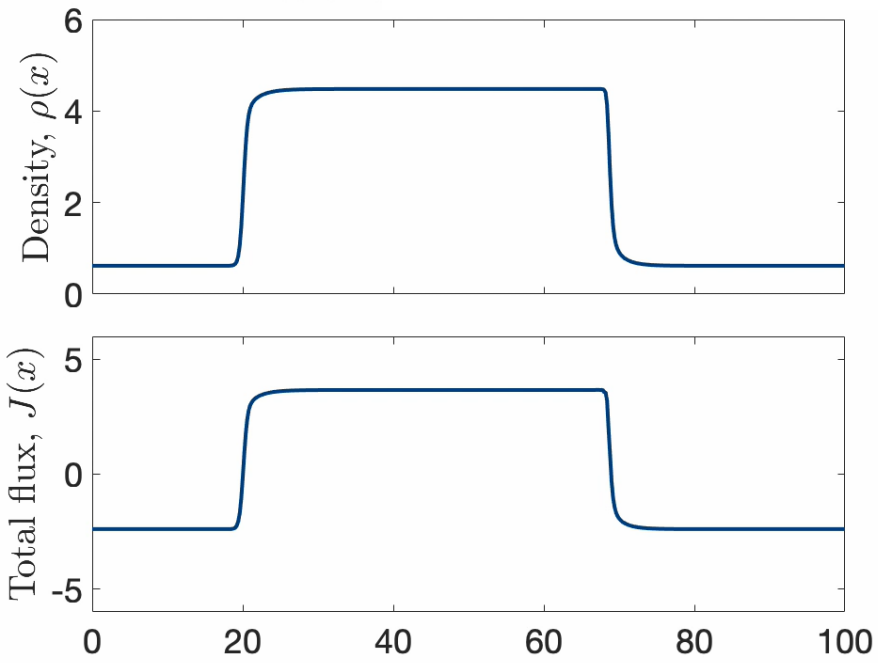






Colour wheel used to determine direction of velocity

Bertrand & Lee (2022) PRR



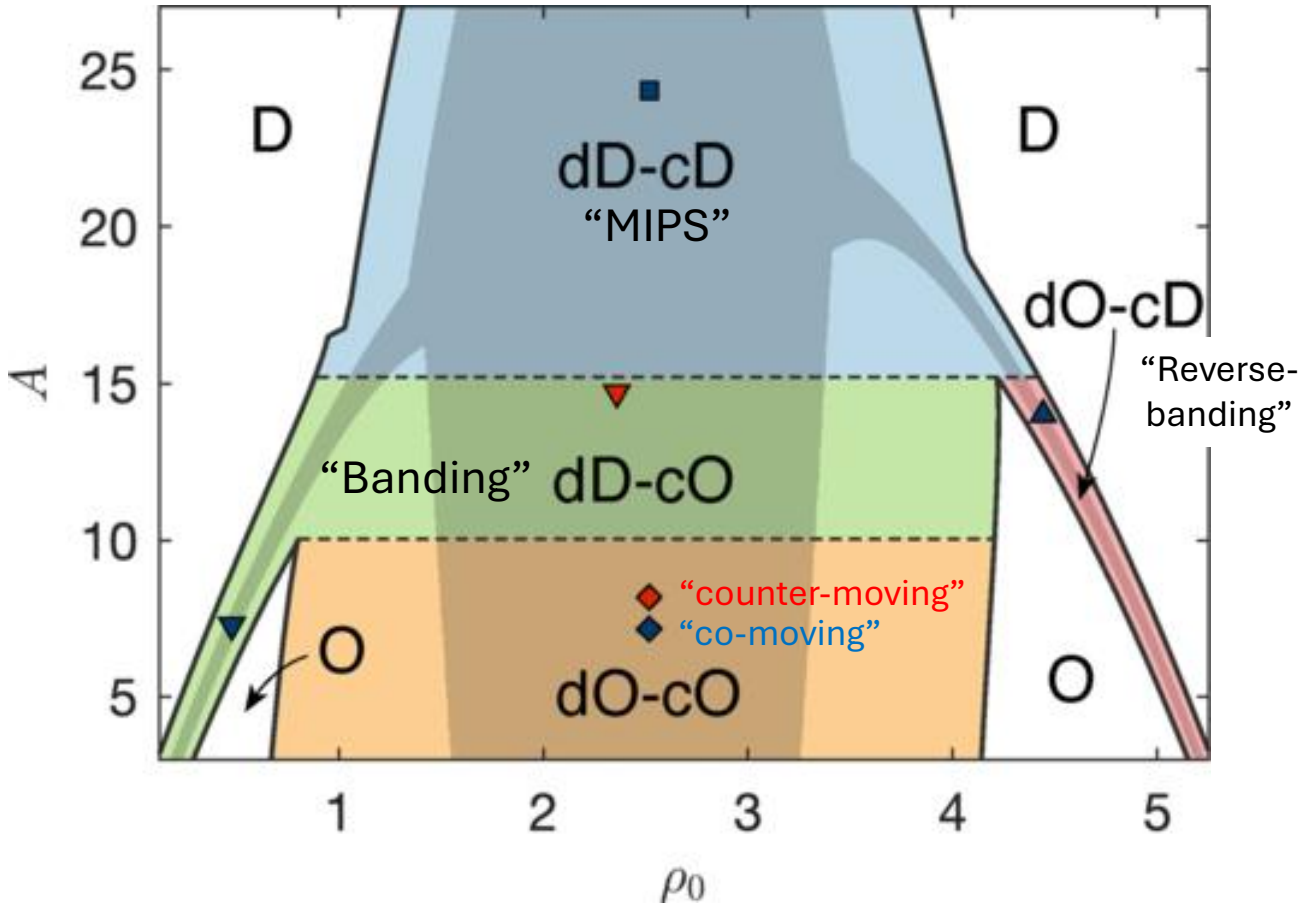
Co-existence of counter-moving phases emerges if we combine “MIPS” with “Reverse banding”

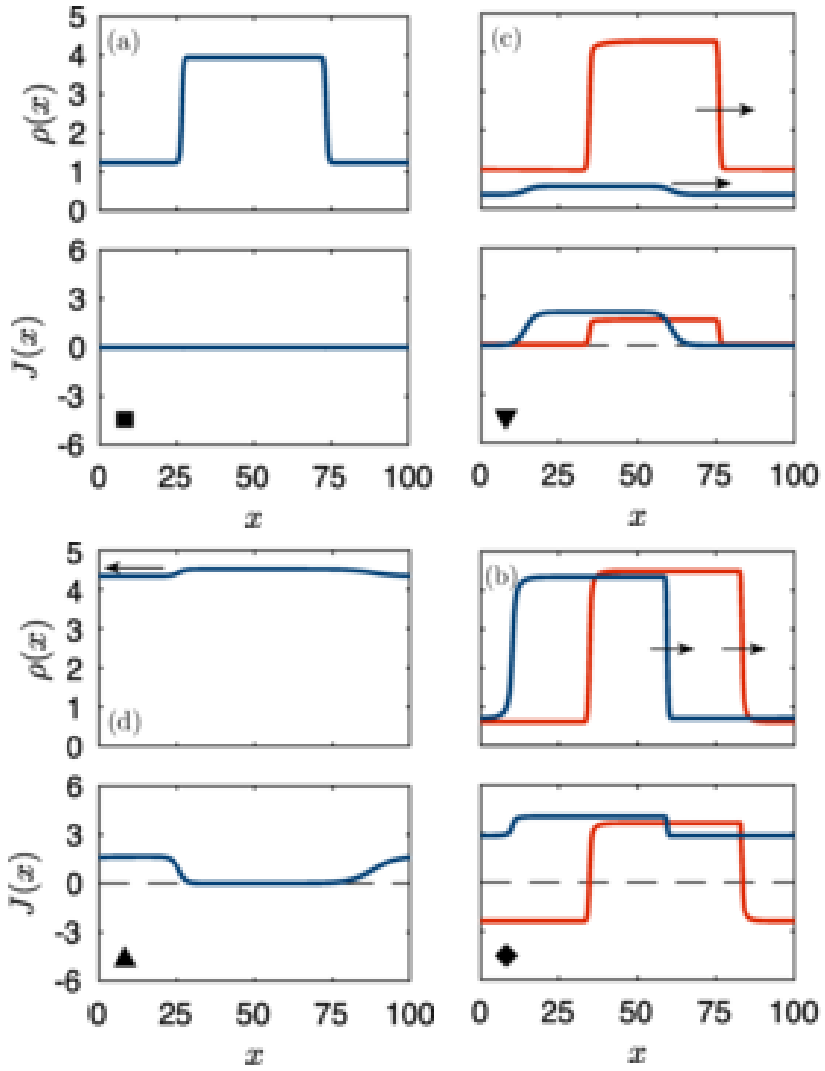
Resulting phase diagram

$$\kappa(\rho) = 140 - 145\rho + 30\rho^2$$

$$F(\rho, m) = \left(-A + 18\rho - \frac{10}{3}\rho^2\right)m - \frac{1}{2}m^3$$

Shaded: instability (~ spinodal decomposition)
Black line: phase boundary (~ binodal)



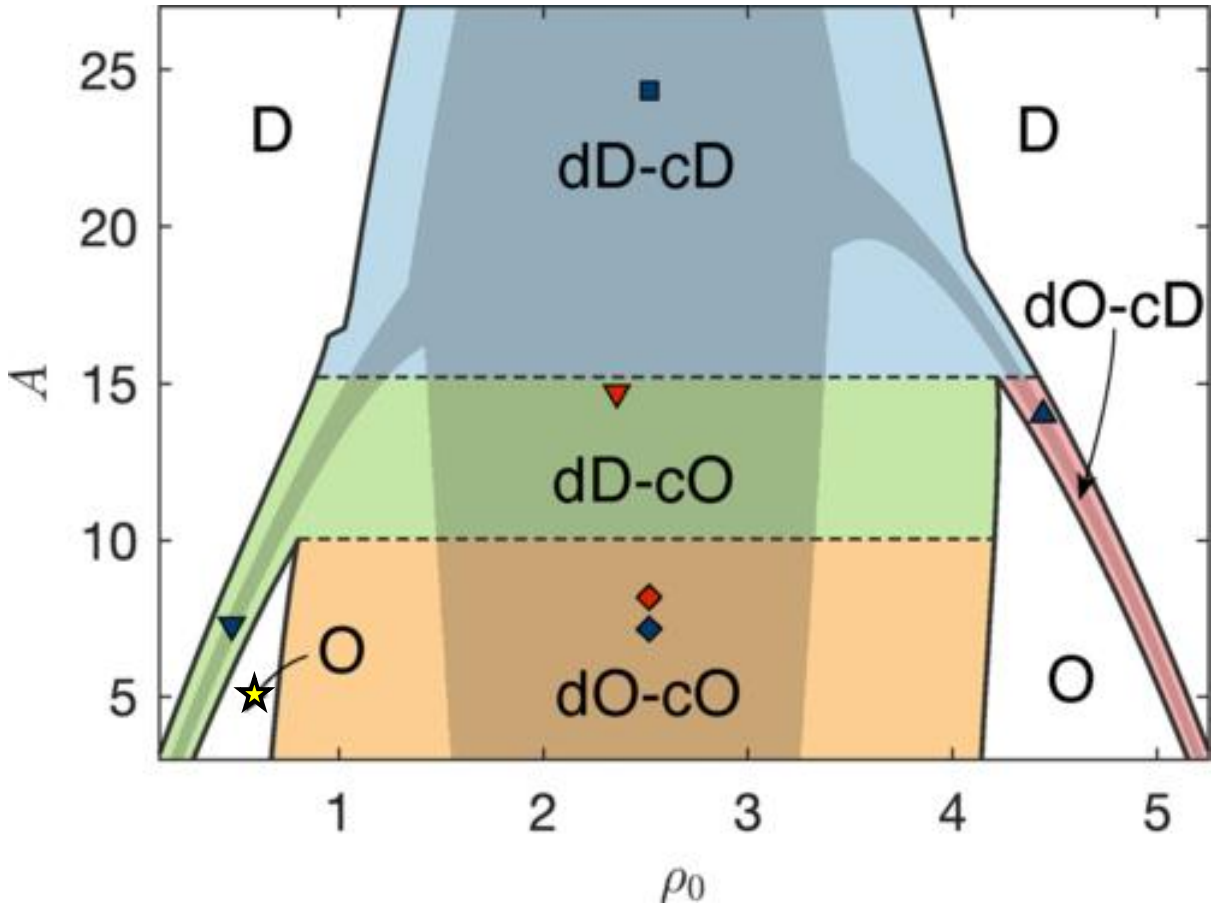


Bertrand & Lee (2022) Physical Review Research

Counterpropagating drops need not grow ballistically



John-Antonio Argyriadis



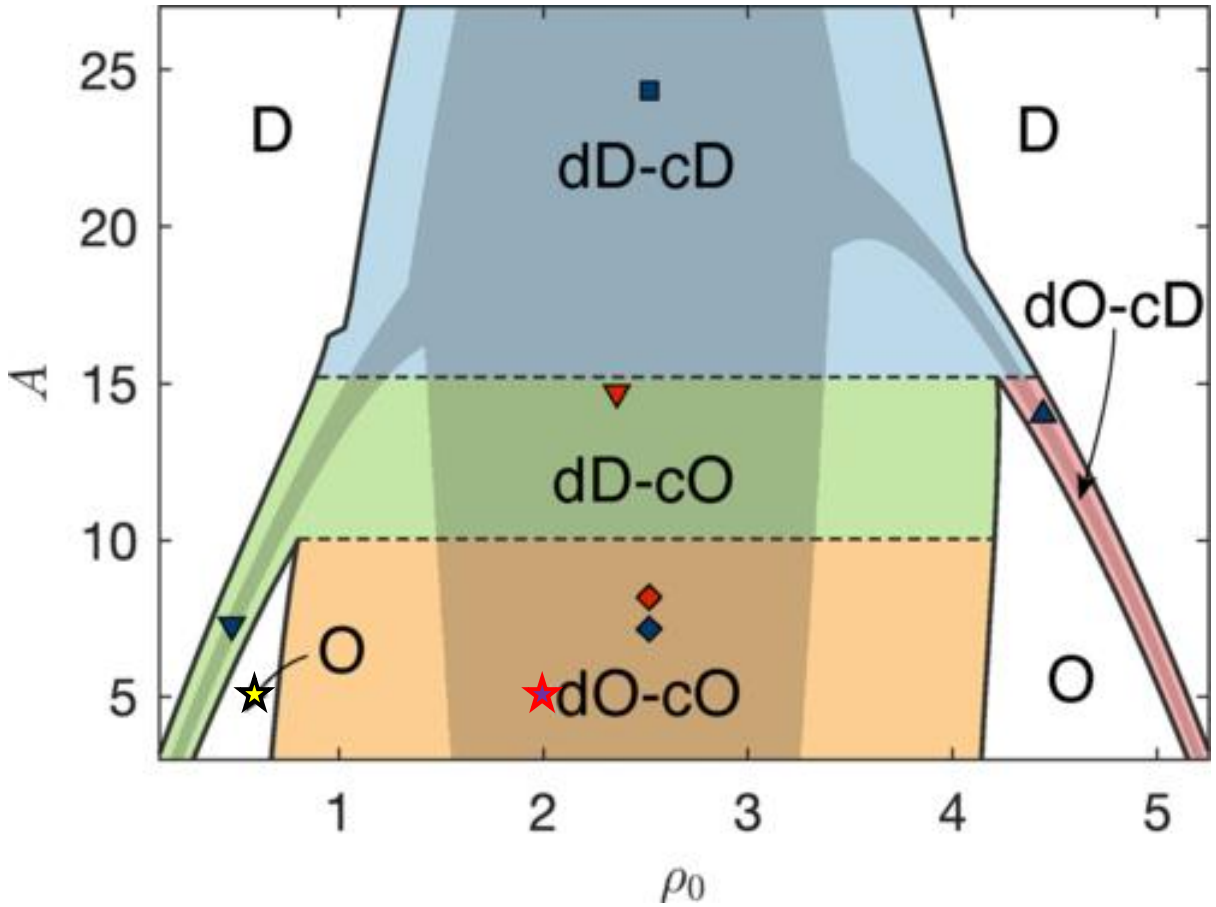
Drops added to the bath ☆

Bertrand & Lee (2022) Physical Review Research

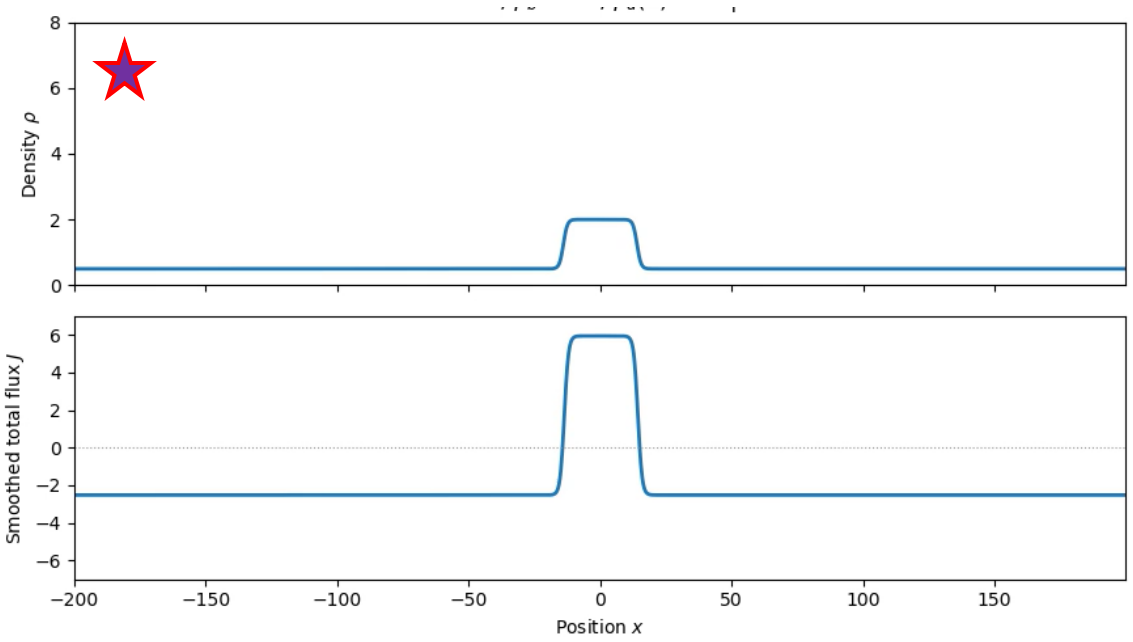
Counterpropagating drops need not grow ballistically



John-Antonio Argyriadis



Drops added to the bath ★

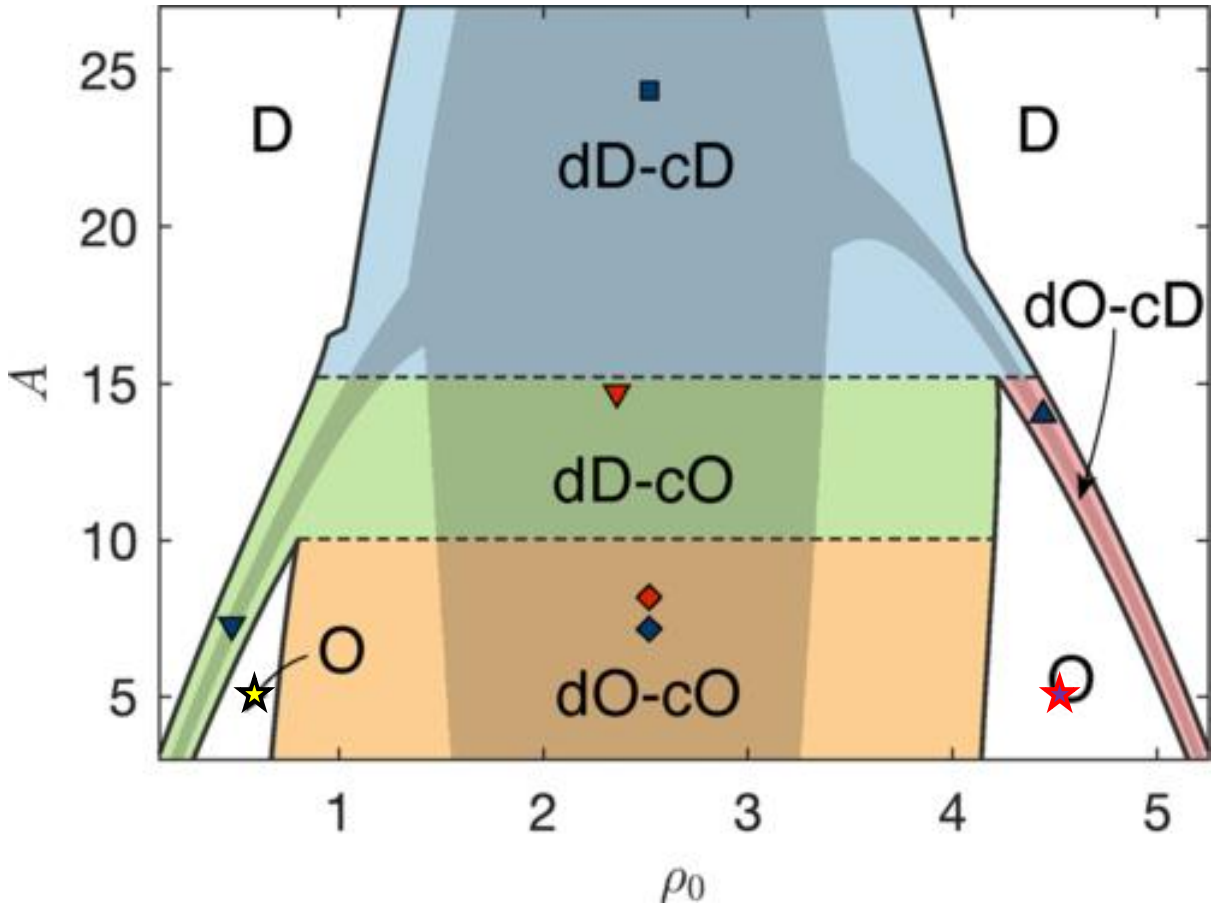


Bertrand & Lee (2022) Physical Review Research

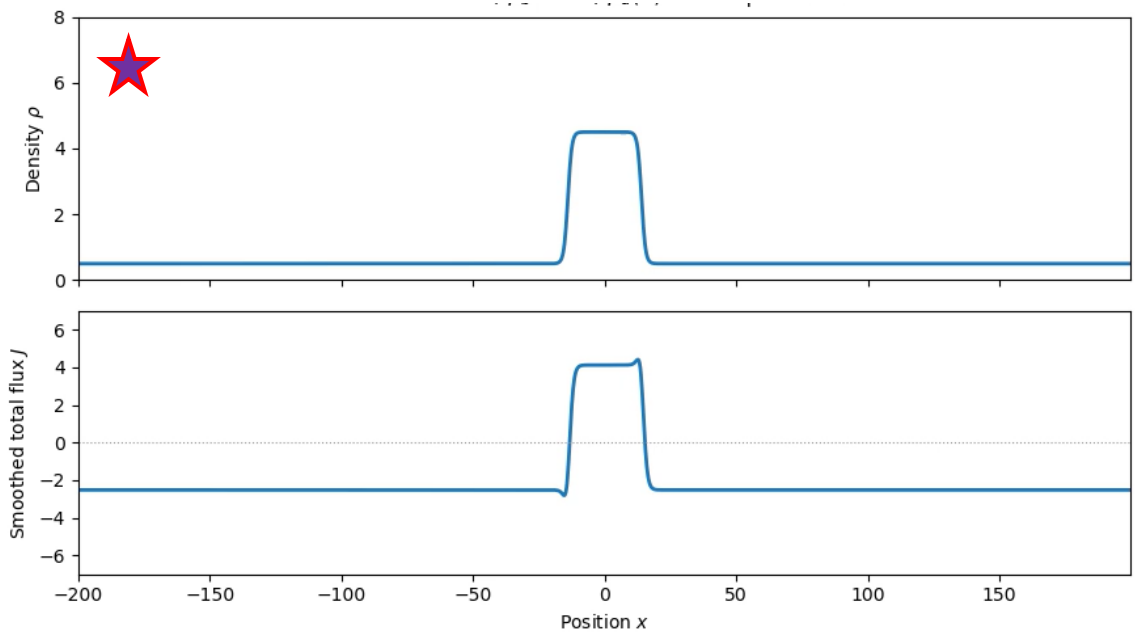
Counterpropagating drops need not grow ballistically



John-Antonio Argyriadis



Drops added to the bath ★



Bertrand & Lee (2022) Physical Review Research

What's the structural difference?

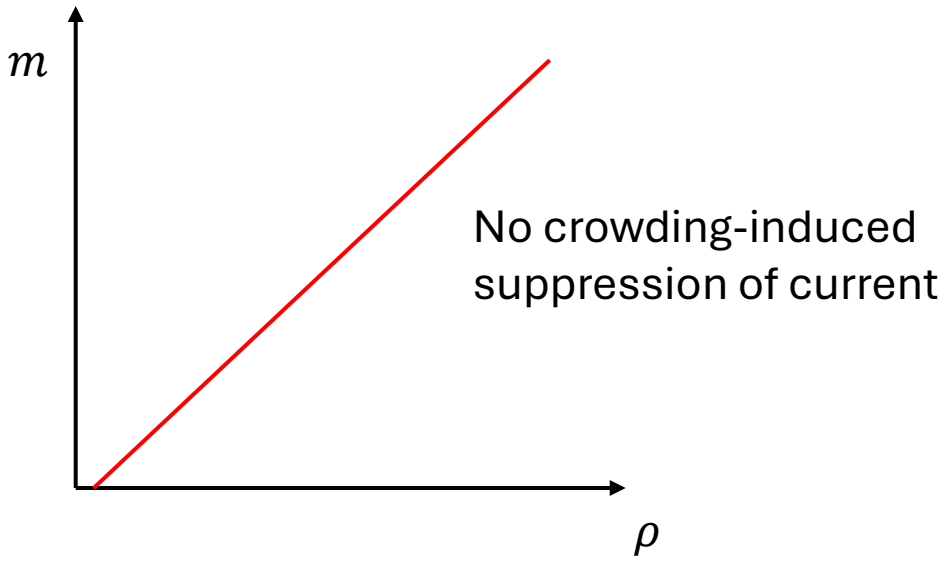
- Benvegnen et al (2023) PRL:

$$F_{\mathrm{B}}(\rho, m) = 2\rho \sinh\left(\frac{\beta m}{\rho}\right) - 2m \cosh\left(\frac{\beta m}{\rho}\right)$$

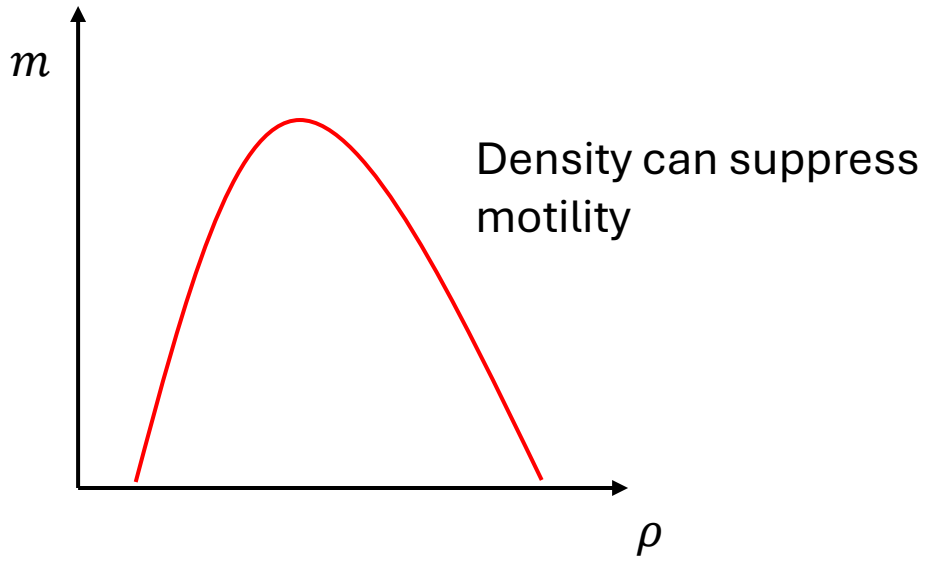
- Bertrand & Lee (2022) PRR:

$$F_{\mathrm{BL}}(\rho, m) = \left(-A + 18\rho - \frac{10}{3}\rho^2\right) m - \frac{1}{2}m^3$$

- Loci of $F_{\mathrm{B}} = 0$



- Loci of $F_{\mathrm{BL}} = 0$

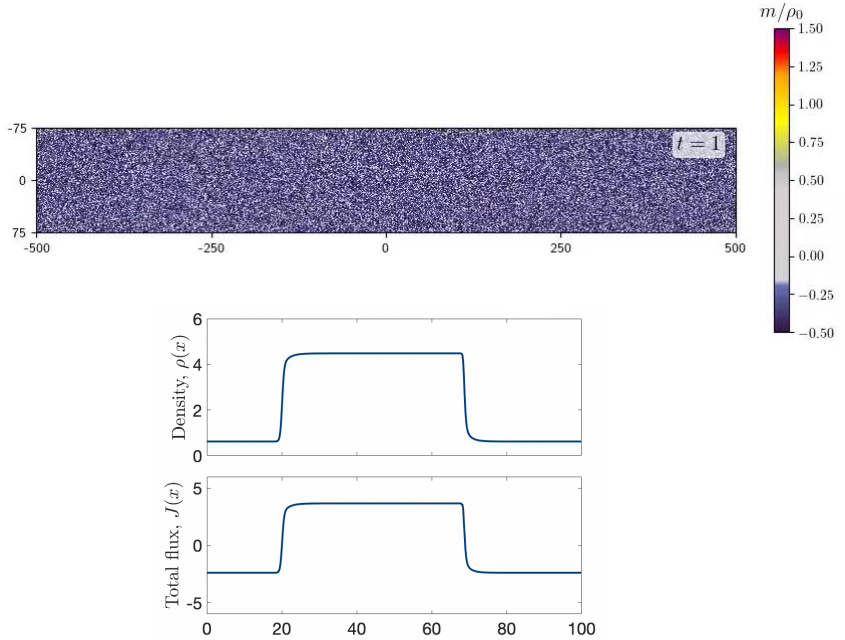


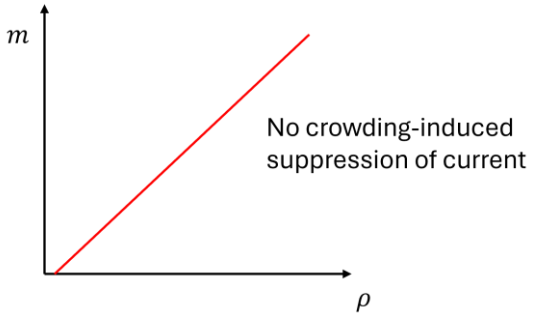
Back to our key question

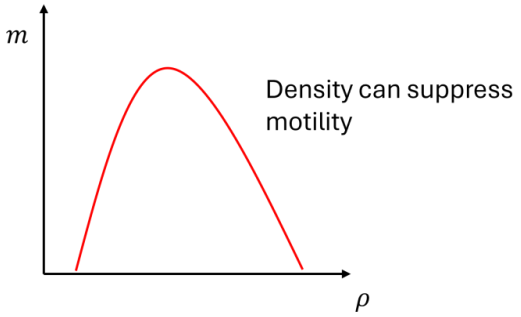
- The Ising model has an ordered phase (for $D \geq 2$)
- Does the ordered phase exist in the “active” Ising model?
- Answer: Yes, in some parameter regime.

Summary

- **Benvegnen et al. (2023):** Ordered phase of active Ising model (AIM) is metastable because rare counterpropagating droplets can nucleate and then grow ballistically
- **Ballistic expansion is not generic:** Counterpropagating dense drops need not expand, as illustrated in a different version of AIM
- **Key difference is density-dependent motility:** Current increases monotonically with density vs. crowding induced current suppression
- **Conclusion:** The active Ising model can support a stable ordered phase







Other Directions: Critical order-disorder transitions

1. Critical active Ising model — conserved density
   - Matt Wong's talk on Tuesday [Wong & Lee (2026) PRE 114, L022103]
2. Malthusian active Ising model — non-conserved density
   - Same as equilibrium Ising (Model A) – WF FP [Bassler & Schmittmann, PRL 73, 3343 (1994)]
   - New multicritical behaviour at a Lifshitz point [Legrand & Lee (2026) Newton 2, 100563]
3. Incompressible active Ising model
   - Dipolar Ising universality: exact exponents in 3D, two-loop results in 2D [Chen, Lee & Toner (2026) PRE 113, 054142]


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Thibault Bertrand
Maths, Imperial College



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