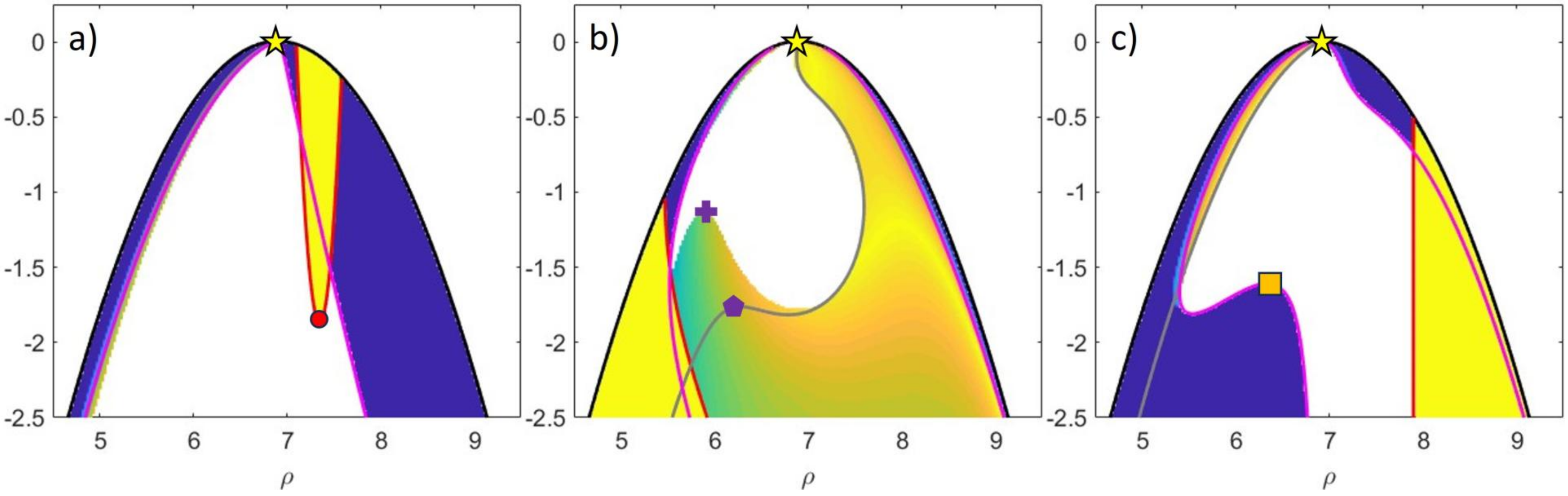

3rd London Workshop on
Active Matter
November 20, 2025

Diversity of critical phenomena in the ordered phase of polar active fluids

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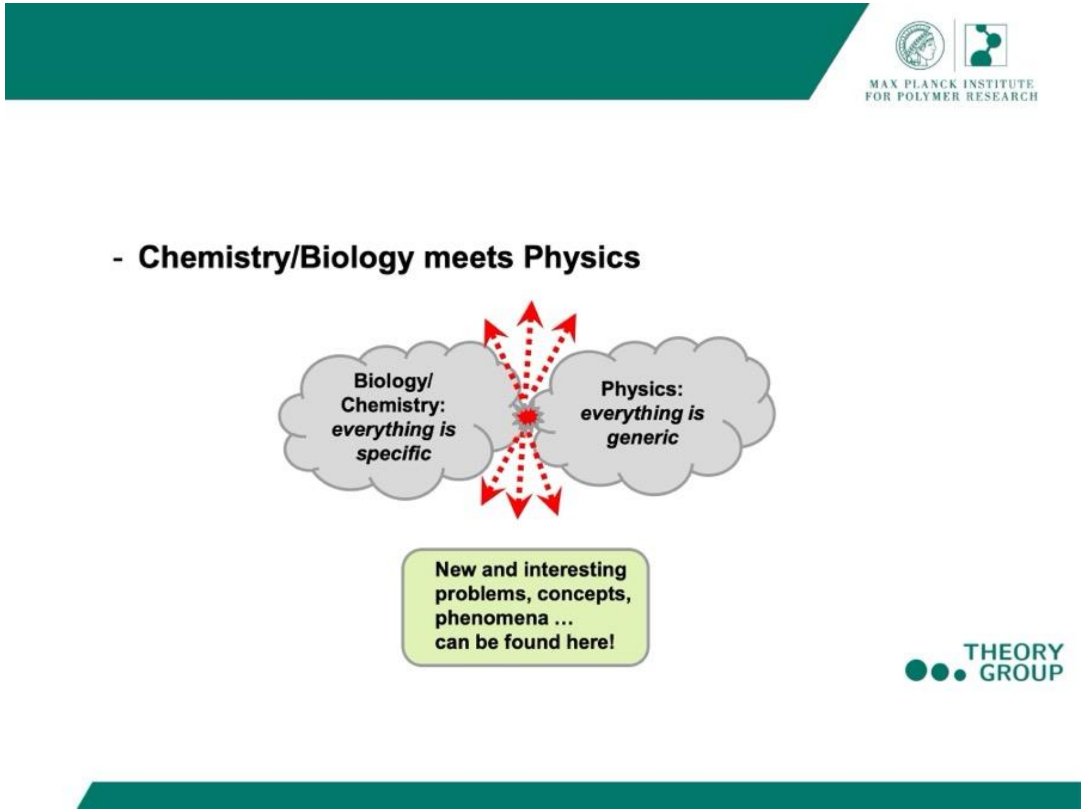
IMPERIAL

Universality in Biology

Research philosophy

A slide from Kurt Kremer (MPI-Polymer Research)



MAX PLANCK INSTITUTE
FOR POLYMER RESEARCH

- **Chemistry/Biology meets Physics**

Biology/ Chemistry: ***everything is specific***

Physics: ***everything is generic***

New and interesting problems, concepts, phenomena …
can be found here!

THEORY GROUP

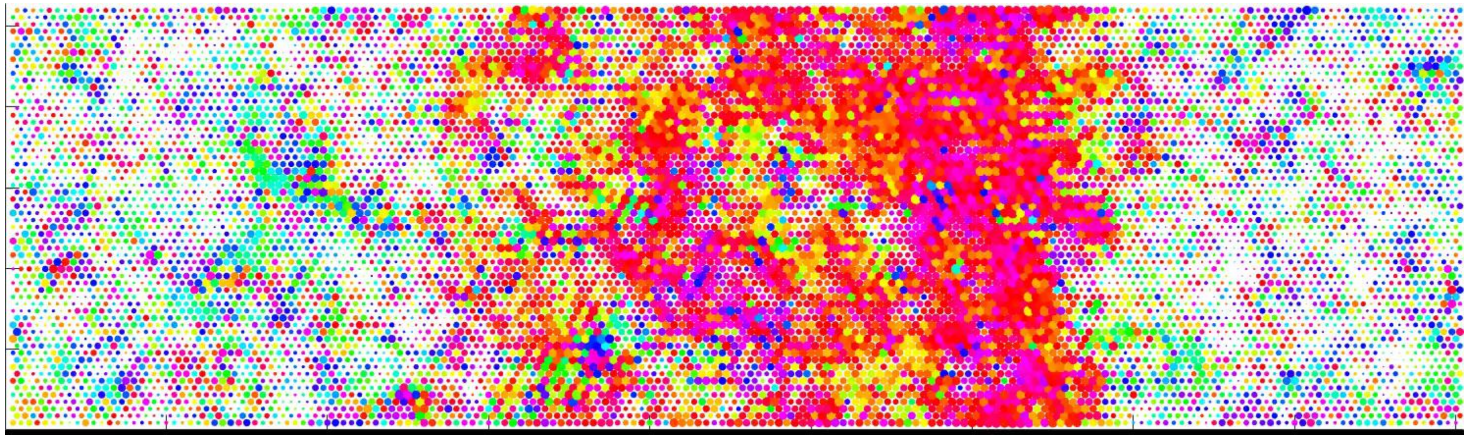
- Generic ↔ universality
- Ex: Navier-Stokes equations
  - Applicable to diverse systems: blood, river, magma, atmosphere,…
- Many physical phenomenon can be partitioned into diverse universality classes
- Perspective: the same applies to biology

Plan

1. Polar active fluids (PAFs)
2. What is a critical point?
3. Critical phenomena in the ordered phase of PAFs
4. Summary & Outlook

POLAR ACTIVE FLUIDS

Colour wheel used to determine direction of velocity



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Universal hydrodynamic model

- Hydrodynamic variables: density ρ and momentum $\vec{g}$

- Conservation law: Mass conservation

$$\boxed{\begin{aligned} \partial_t \rho + \nabla \cdot \mathbf{g} &= 0 \\ \partial_t \mathbf{g} &= \mathbf{F} \end{aligned}}$$

Symmetries

$$\boxed{\begin{aligned}\partial_t \rho + \nabla \cdot \mathbf{g} &= 0 \\ \partial_t \mathbf{g} &= \mathbf{F}\end{aligned}}$$

- What is the force **F**?

- Starting with symmetries:
  - Temporal invariance: **F** does not depend on time
  - Translational invariance: **F** does not depend on position **r**
  - Rotational invariance: **F** does not depend on a particular direction
  - Chiral (parity) invariance: **F** is not right-handed or left-handed

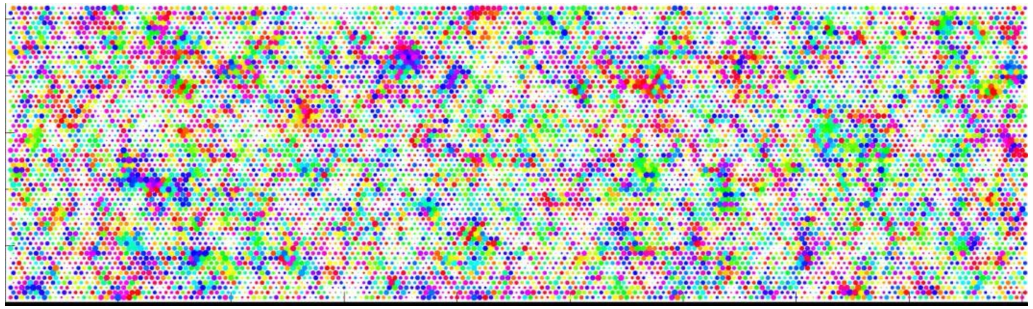
- Universal hydrodynamic EOM for generic polar active fluids (Toner-Tu EOM):

$$\partial_t \rho + \nabla \cdot \mathbf{g} = 0$$

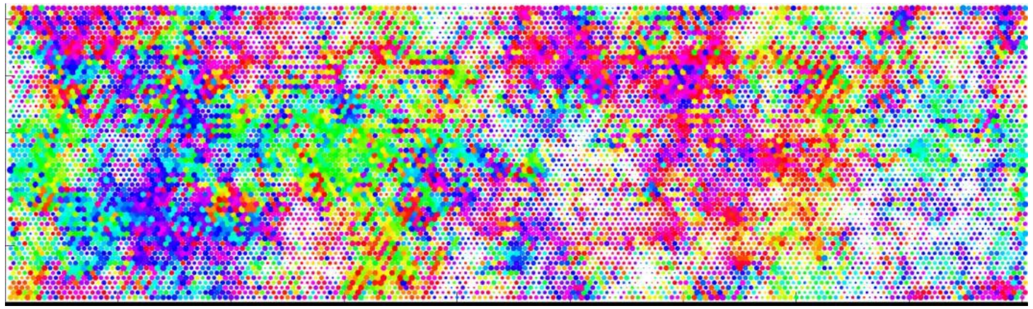
$$\partial_t \mathbf{g} + \lambda_1 (\mathbf{g} \cdot \nabla)\mathbf{g} + \lambda_2 (\nabla \cdot \mathbf{g})\mathbf{g} + \lambda_3 \nabla g^2 = U(\rho, \mathbf{g})\mathbf{g} - \kappa_1 \nabla \rho - \kappa_2 \mathbf{g}(\mathbf{g} \cdot \nabla \rho) + \mu \nabla^2 \mathbf{g} + \underset{\substack{\uparrow \\ \text{Gaussian noise terms}}}{\mathbf{f}} + \cdots$$

Colour wheel used to determine direction of velocity

Disorder



Near criticality



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WHAT IS A CRITICAL POINT?

Thermal phase separation

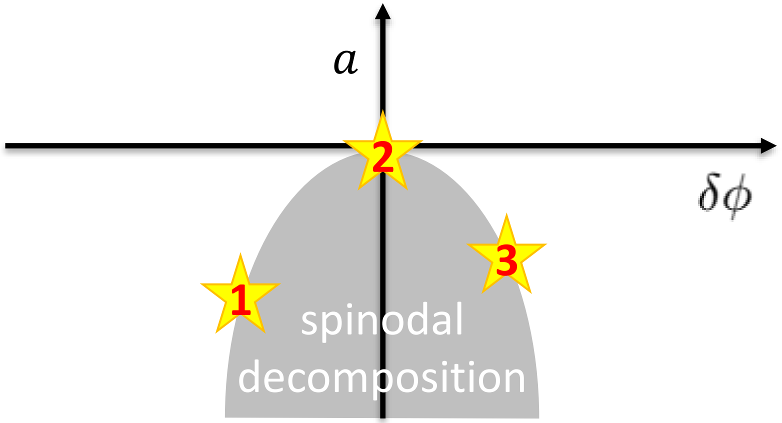
Cahn-Hilliar equation:

$$\boxed{\partial_t \phi = M\nabla^2\big(a\phi + b\phi^3 - \kappa\nabla^2\phi\big).}$$

- Linear stability

$$\phi(\mathbf{r},t) = \phi_0 + \delta\phi(\mathbf{r},t), \qquad \delta\phi = \delta\phi \mathrm{e}^{st-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \xrightarrow{\text{CH model eqn}} s = -Mk^2\left[f''(\phi_0) + \kappa k^2\right]$$

- Instability region (spinodal decomposition)



- Where is the critical point?

- Criticality criteria: $s|_{\delta\phi} = 0 \quad , \quad \left.\dfrac{\partial s}{\partial\delta\phi}\right|_{\delta\phi} = 0$

Thermal phase separation

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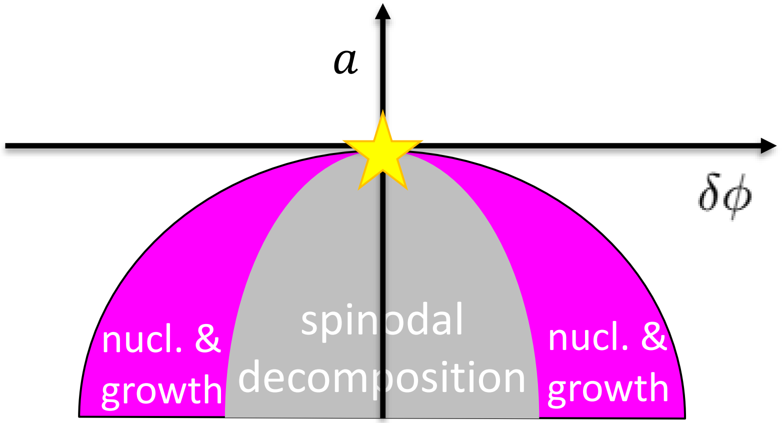
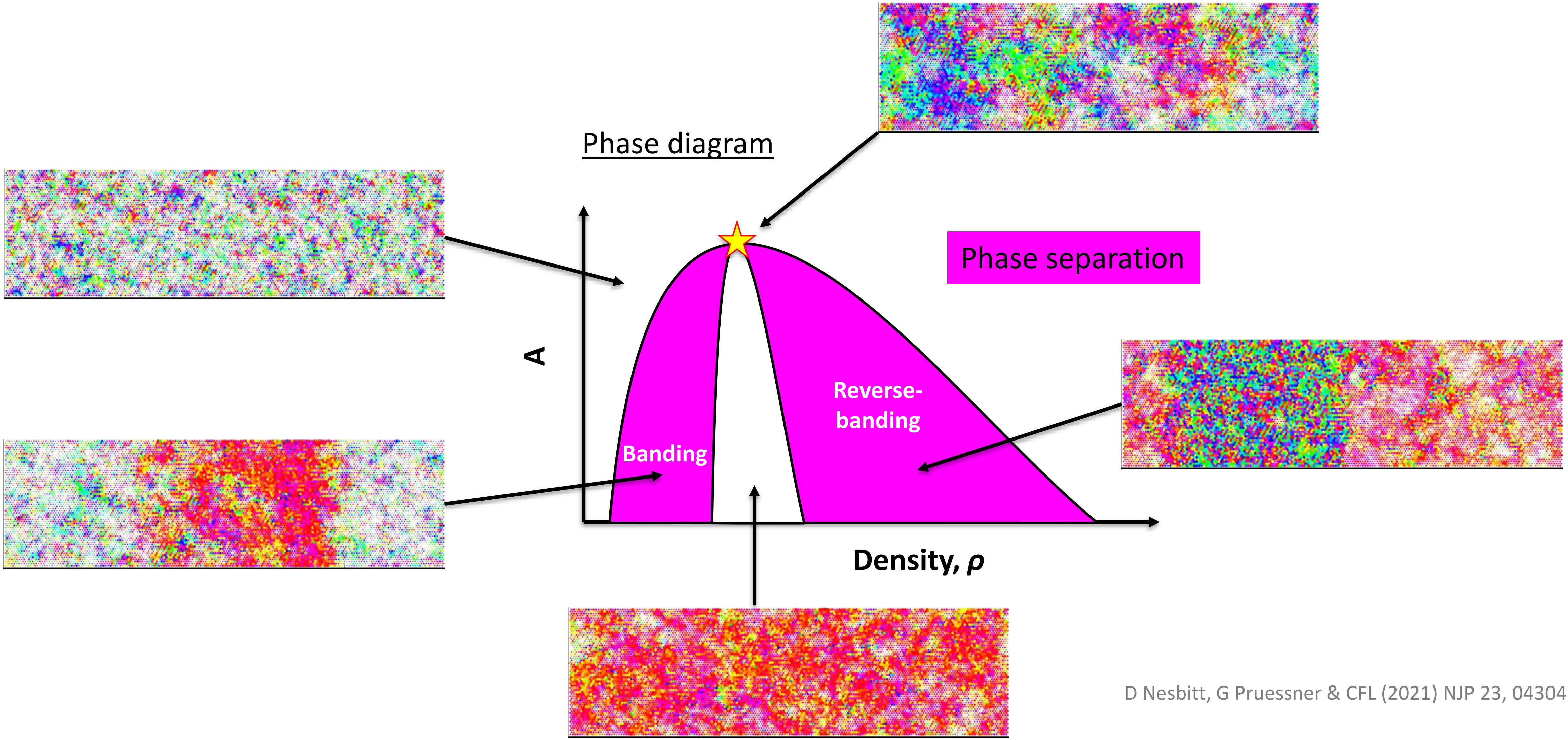
- Where is the critical point?



- Criticality criteria: $s|_{\delta\phi} = 0 \quad , \quad \left.\frac{\partial s}{\partial \delta\phi}\right|_{\delta\phi} = 0$

Ordered-disordered critical point

Colour wheel used to determine direction of velocity



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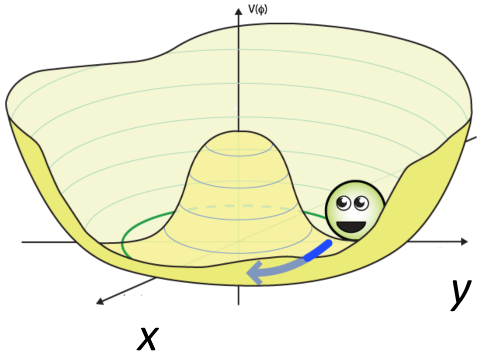
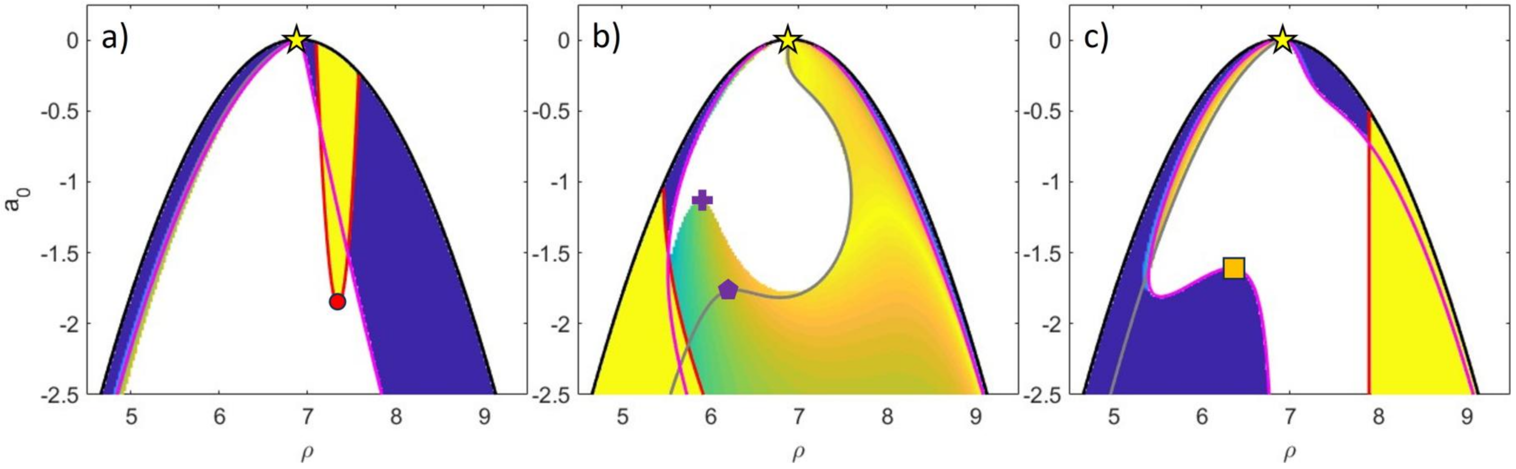


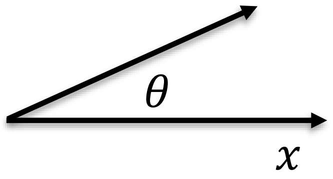
Fig from QuantumDiaries.org



P Jentsch & CFL, in preparation

CRITICAL PHENOMENA IN THE ORDERED PHASE OF PAFS

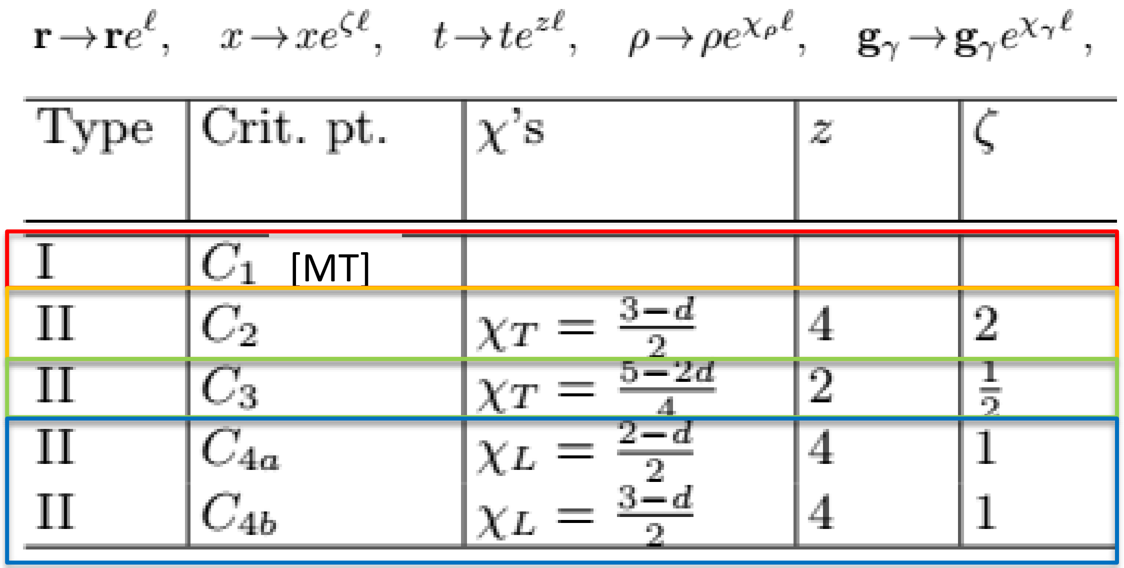
Linear stability in the ordered phase



$$\rho(t,\mathbf{r}) = \rho_0 + \delta\rho(t,\mathbf{r}) \quad , \quad \mathbf{g}(t,\mathbf{r}) = \mathbf{g}_0 + \mathbf{g}_x(t,\mathbf{r}) + \mathbf{g}_\perp(t,\mathbf{r})\,, \qquad \delta\rho(t,\mathbf{r}) = \delta\rho \mathrm{e}^{st-\mathrm{i}\mathbf{q}\cdot\mathbf{r}}, \text{ etc}$$

$$s \xrightarrow{q\to 0} \begin{cases} E_0 & = -\beta g_0^2 \\ E_\pm & = \mathrm{i}qA_1(\theta) \pm \mathrm{i}q\boxed{\sqrt{A_2(\theta)}} - \boxed{q^2 B_1(\theta) \pm q^2 \frac{B_2(\theta)}{\sqrt{A_2(\theta)}}} \\ E_T & = -\mathrm{i}\lambda_1 g_0 q_x - \boxed{\mu_x} q_x^2 - \boxed{\mu_1} q_\perp^2 \end{cases}$$

Type I ~ q
Type II ~ q^2



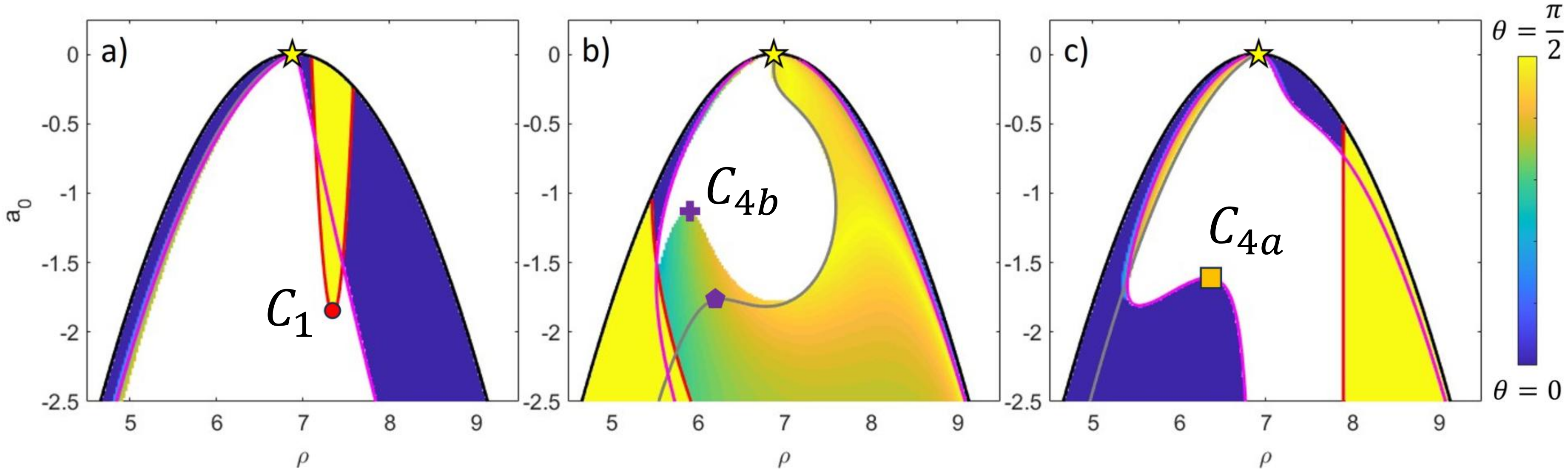
$$\mathbf{r}\to\mathbf{r}e^{\ell}, \quad x\to xe^{\zeta\ell}, \quad t\to te^{z\ell}, \quad \rho\to\rho e^{\chi_\rho\ell}, \quad \mathbf{g}_\gamma\to\mathbf{g}_\gamma e^{\chi_\gamma\ell},$$

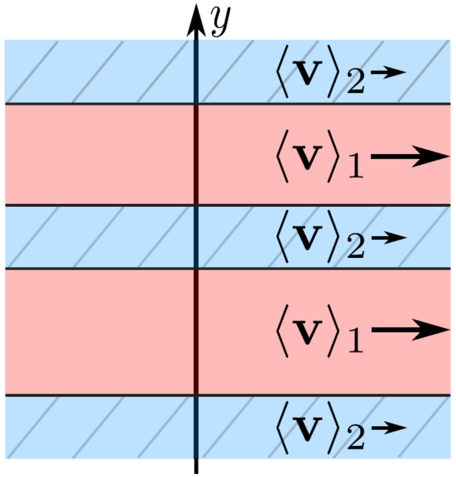
Type	Crit. pt.	χ's	z	ζ
I	C_1 [MT]			
II	C_2	$\chi_T = \frac{3-d}{2}$	4	2
II	C_3	$\chi_T = \frac{5-2d}{4}$	2	$\frac{1}{2}$
II	C_{4a}	$\chi_L = \frac{2-d}{2}$	4	1
II	C_{4b}	$\chi_L = \frac{3-d}{2}$	4	1

P Jentsch & CFL, in preparation

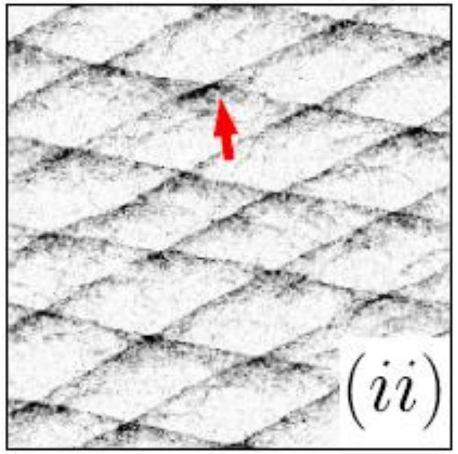
[MT] M. Miller and J. Toner, Spinodal decomposition and phase separation in polar active matter, Physical Review E **109**, 034606 (2024).

Hydrodynamic models realising these novel CPs

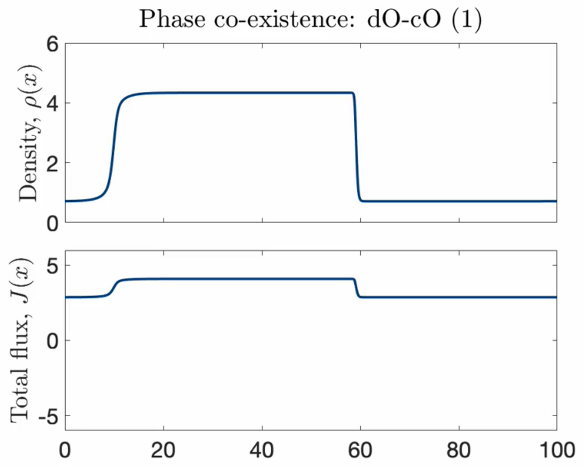




Miller & Toner (2024) PRE



Kürsten & Ihle (2020) PRL



Bertrand & CFL (2022) PRR

Summary

1. Use symmetries and conservation law to derive the universal model of polar active fluids (PAFs)
2. Criteria for instability-induced critical behaviour
3. Comprehensive list of critical points in the ordered phase of PAFs

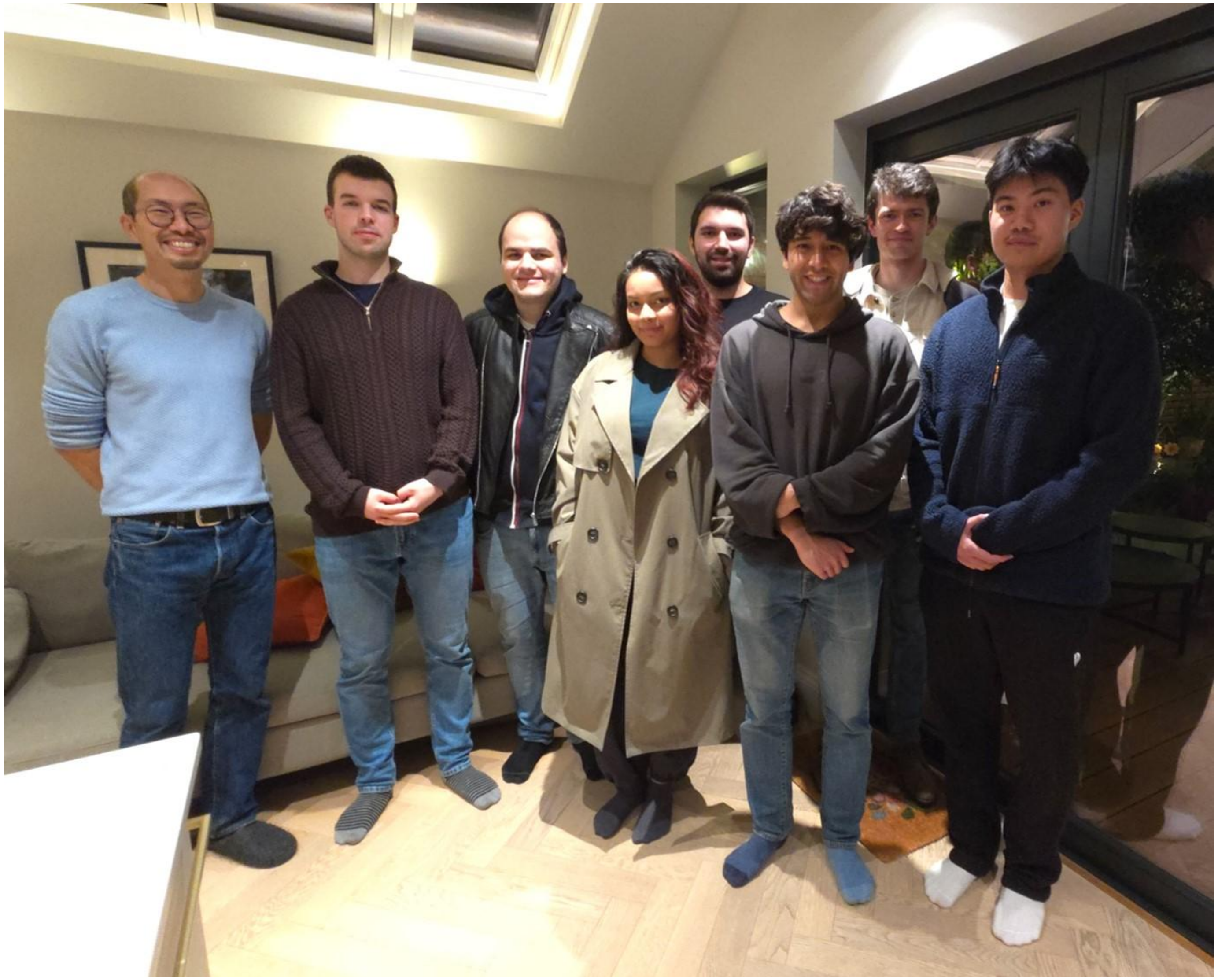
Outlook 1

Now that we know where to dig, how do we actually start digging?

Gold standard: Renormalisation group analysis (Wilsonian, field-theoretic, functional)

Acknowledgement

Patrick Jentsch (EMBL)



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