Soft Condensed Matter Seminar
Harvard University
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Diversity of Scale-Invariant Structures in Adaptive Swarms

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IMPERIAL

Universality in Biology

Q: How do adaptive swarms spontaneously generate scale-free structure?

1. Swarms as active fluids
2. How adaptive rules explores diverse phases
3. When soft modes exhibit universality
4. A periodic table of nonequilibrium universality classes
5. Grand challenge: generic 2D compressible flocks

1. Swarms → Hydrodynamics

Swarms

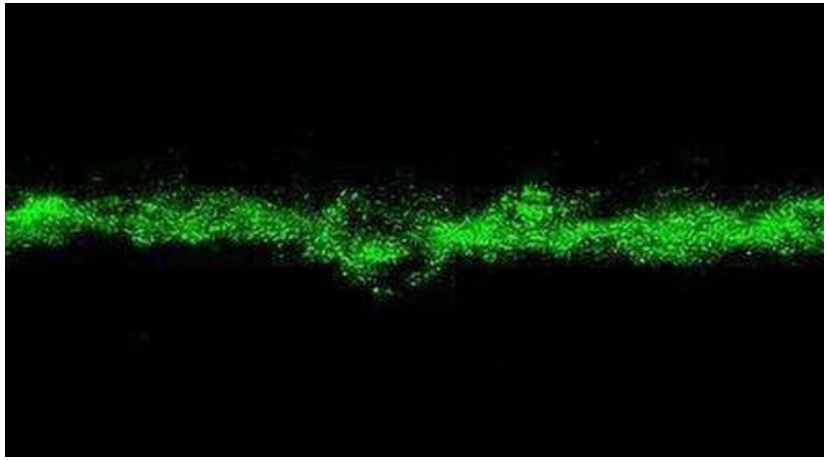
Locust swarm (~10 trillions)

Wildebeest (2 millions)

Drones (~10 millions)

Mexican free-tailed bat (20 millions)

Bacterial colony (1 trillions)



Sujit Datta Lab

How can we study these system theoretically ?

- **Large scales and large numbers → hydrodynamic description**
  capturing the long-wavelength, long-time dynamics of coarse-grained fields
- **Absence of fixed neighbours → fluid-like behaviour**

  Navier-Stokes → two hydrodynamic fields

Hydrodynamic model

- Hydrodynamic variables: density ρ and momentum $\vec{g}$

- Conservation law: Mass conservation

$$\boxed{\begin{aligned} \partial_t \rho + \nabla \cdot \mathbf{g} &= 0 \\ \partial_t \mathbf{g} &= \mathbf{F} \end{aligned}}$$

Symmetries

$$\boxed{\begin{aligned} \partial_t \rho + \nabla \cdot \mathbf{g} &= 0 \\ \partial_t \mathbf{g} &= \mathbf{F} \end{aligned}}$$

- What is the force **F**?

- Starting with symmetries:
  - Temporal invariance: **F** does not depend on time
  - Translational invariance: **F** does not depend on position **r**
  - Rotational invariance: **F** does not depend on a particular direction
  - Chiral (parity) invariance: **F** is not right-handed or left-handed

- Universal hydrodynamic EOM for generic polar active fluids (Toner-Tu EOM):

$$\partial_t \rho + \nabla \cdot \mathbf{g} = 0$$

$$\partial_t \mathbf{g} + \lambda_1 (\mathbf{g} \cdot \nabla)\mathbf{g} + \lambda_2 (\nabla \cdot \mathbf{g})\mathbf{g} + \lambda_3 \nabla g^2 = U(\rho, \mathbf{g})\mathbf{g} - \kappa_1 \nabla \rho - \kappa_2 \mathbf{g}(\mathbf{g} \cdot \nabla \rho) + \mu \nabla^2 \mathbf{g} + \mathbf{f} + \cdots$$

Gaussian noise terms

ChatGPT (DALL·E)

2. Adaptive microscopic rules → exploration of hydrodynamic parameter space

What does it mean to our hydrodynamic model?

$$\partial_t \rho + \nabla \cdot \mathbf{g} = 0$$

$$\partial_t \mathbf{g} + \lambda_1 (\mathbf{g} \cdot \nabla)\mathbf{g} + \lambda_2 (\nabla \cdot \mathbf{g})\mathbf{g} + \lambda_3 \nabla g^2 = U(\rho, \mathbf{g})\mathbf{g} - \kappa_1 \nabla \rho - \kappa_2 \mathbf{g}(\mathbf{g} \cdot \nabla \rho) + \mu \nabla^2 \mathbf{g} + \mathbf{f} + \cdots$$

Adaptive swarms can dynamically tune alignment, sensing, and response

→ access distinct regions of phase space

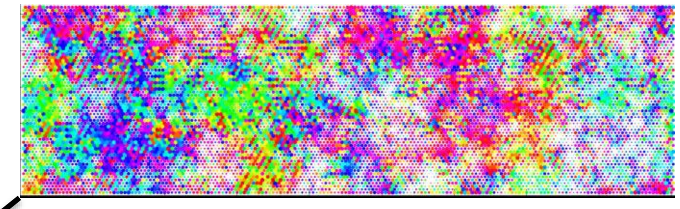
→ reveal new instabilities and universality classes

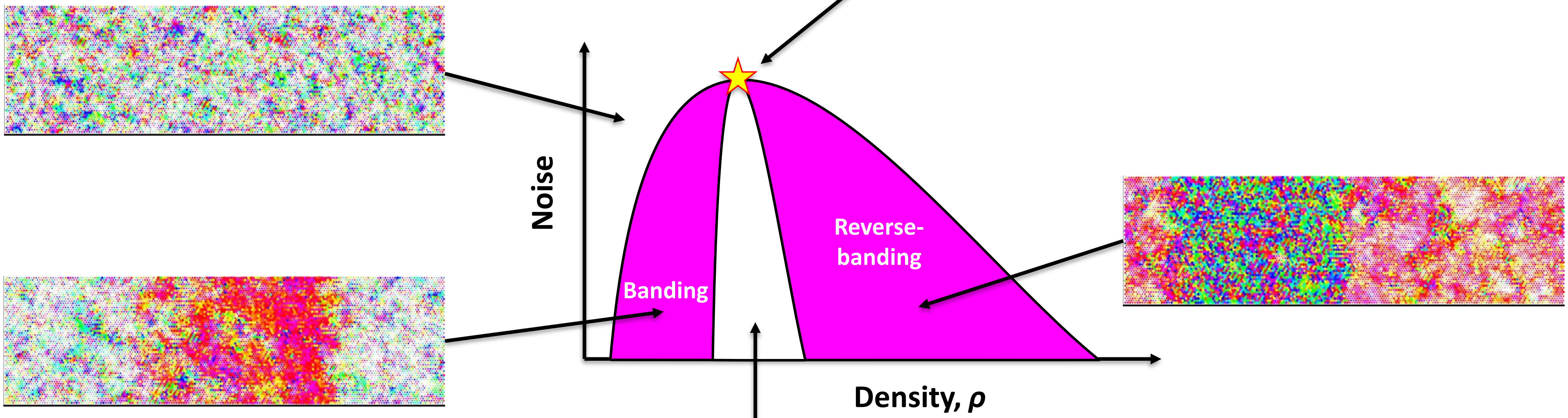
A sample phase diagram

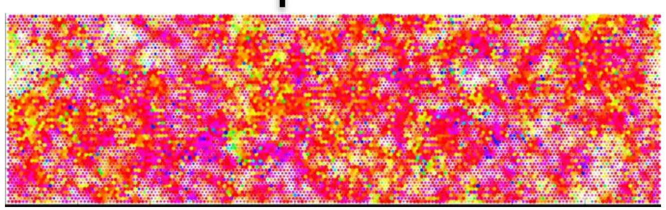
Colour wheel used to determine direction of velocity

Alignment term $\alpha(\rho)$:

$$\alpha(\rho) = A_0 + A_1(\rho - \rho_{\rm ref}) + A_2(\rho - \rho_{\rm ref})^2$$

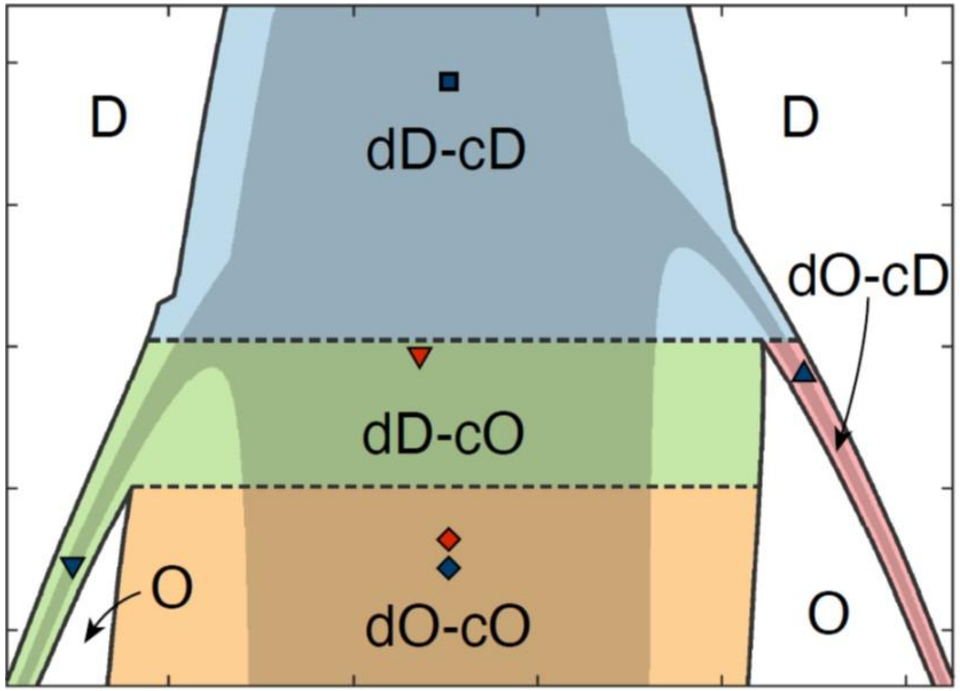




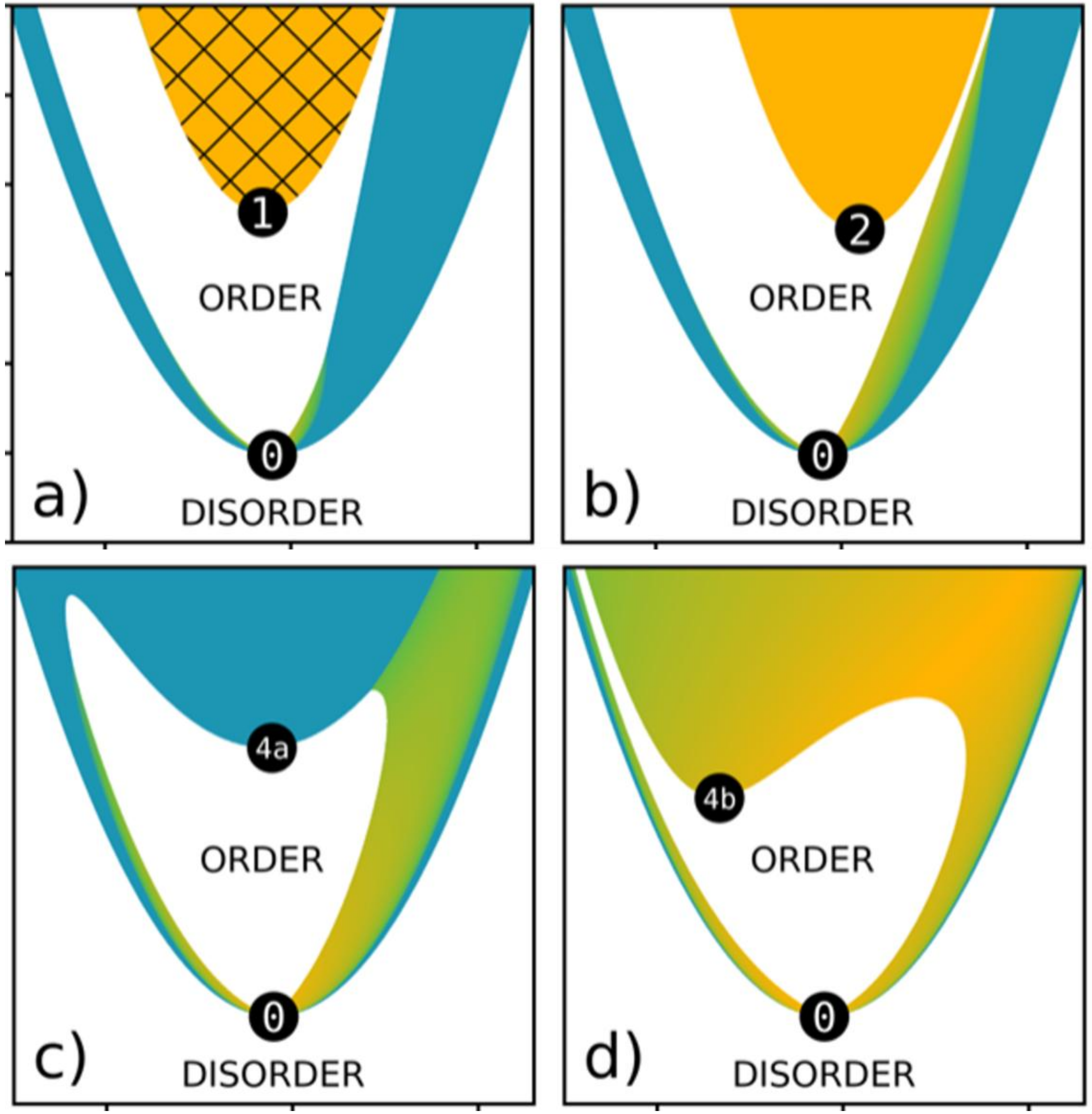


D Nesbitt, G Pruessner & CFL (2021) NJP 23, 043047

More exotic phase diagrams



Bertrand & Lee (2022) Phys Rev Res



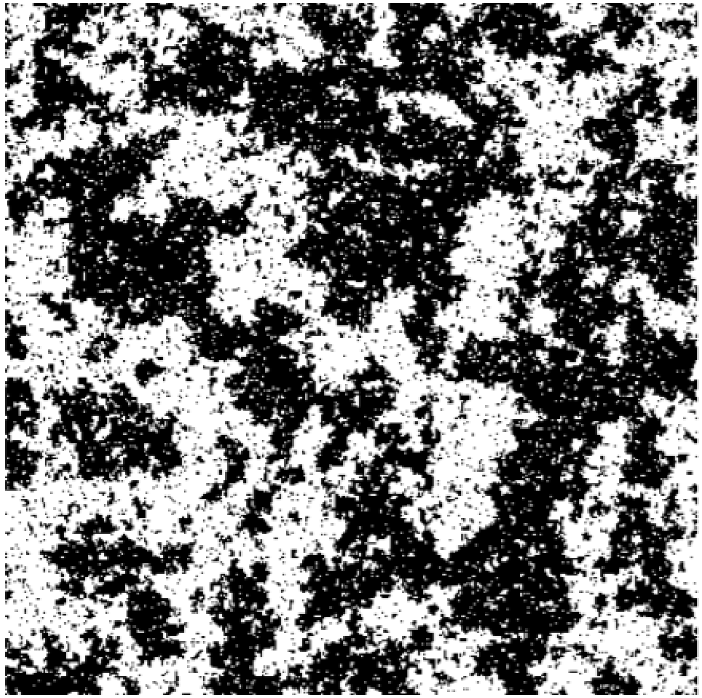
Jentsch & Lee (2025) arXiv

$$\partial_t \phi = \mu \nabla^2 \phi - \beta \phi^3 + f$$

$$x \to e^{\ell} x, \qquad t \to e^{z\ell} t, \qquad \phi(x,t) \to e^{\chi \ell} \phi(x,t).$$

$$\langle \phi(x,t)\, \phi(0,0) \rangle = |x|^{2\chi}\, \mathcal{S}\left(\frac{t}{|x|^z} \right).$$

z : dynamic exponent ; χ : roughness exponent



Rushkin *et al* 2007 *J. Phys. A: Math. Theor.*

3. Scale-invariant structures
→ RG fixed points & universality classes (UCs)

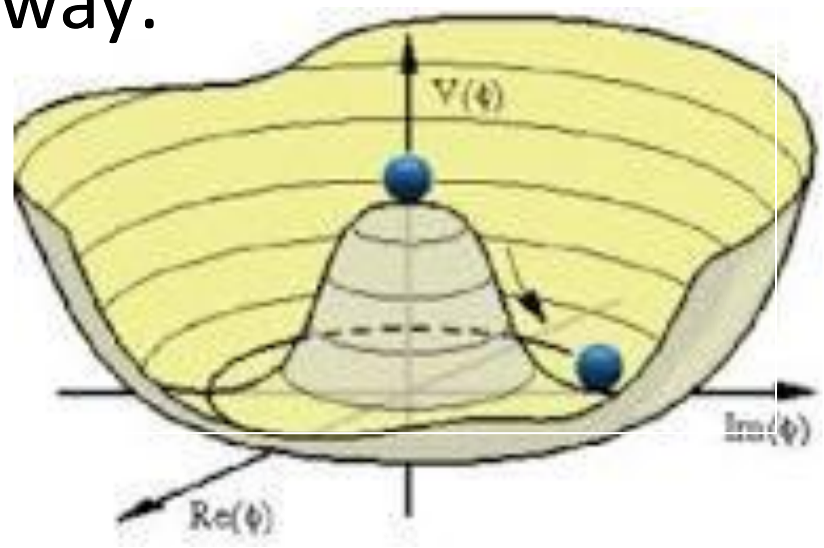
When to expect scale-invariant structures?

Expectation: When EOM of hydrodynamic field(s) become soft (massless, gapless)?

- Example (by fine-tuning): $\partial_t\phi = \mu\nabla^2\phi - \cancel{\alpha\phi} - \beta\phi^3 + f$

Besides fine-tuning, there's a more **general** way:

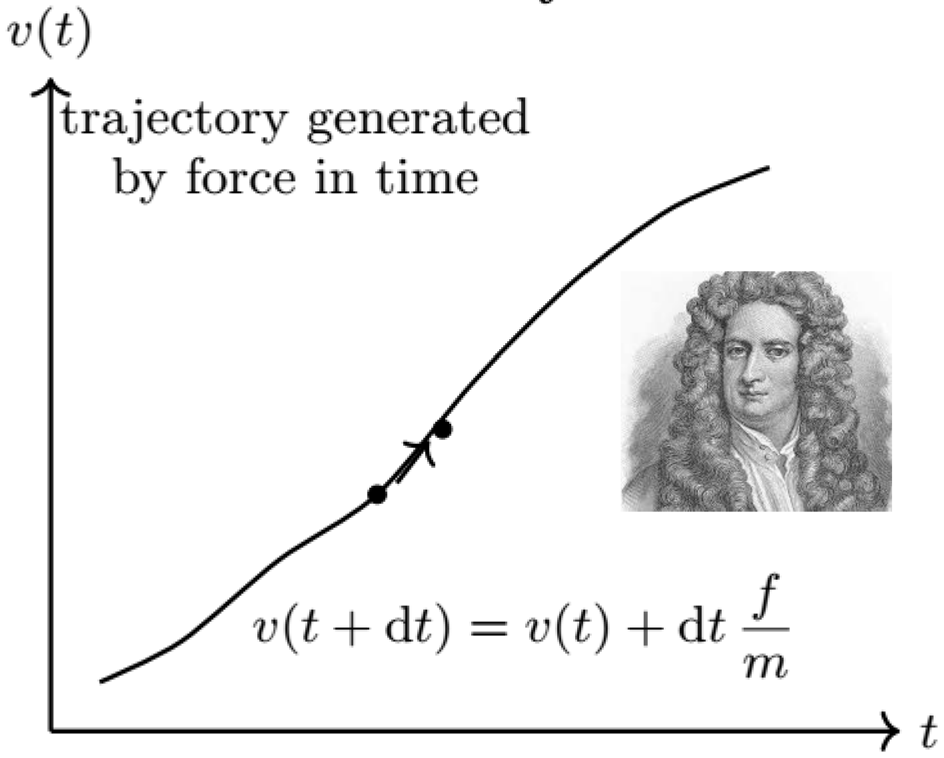
- Continuous symmetry is spontaneously broken (Goldstone modes)



How to go from soft EOM to scale-invariance?

Scale-invariance = Renormalisation Group Fixed Point = Universality Class

Newtonian dynamics



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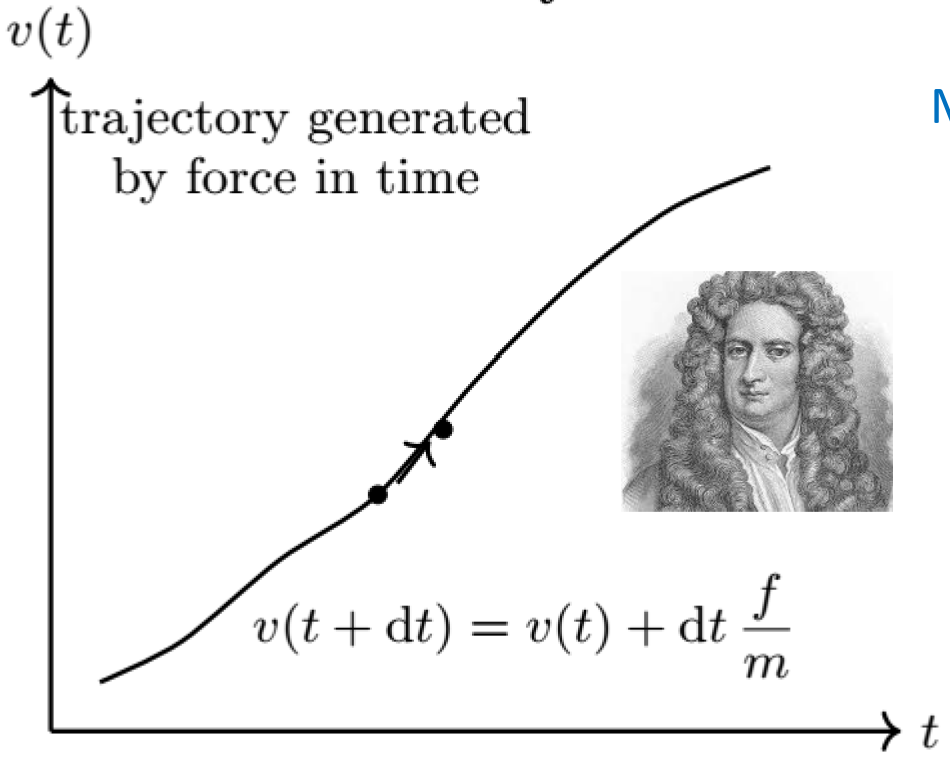

Many-body
problems
+ noise

Fluctuating hydrodynamics and RG

$$\boxed{\partial_t \phi = \mu \nabla^2 \phi - \beta \phi^3 + f}$$

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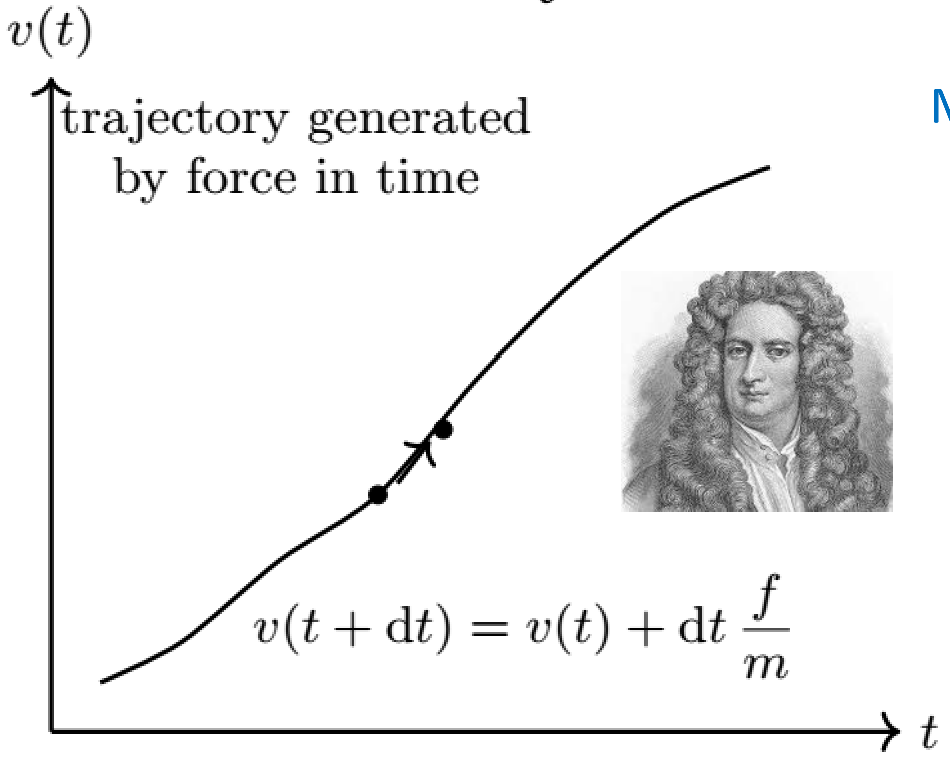

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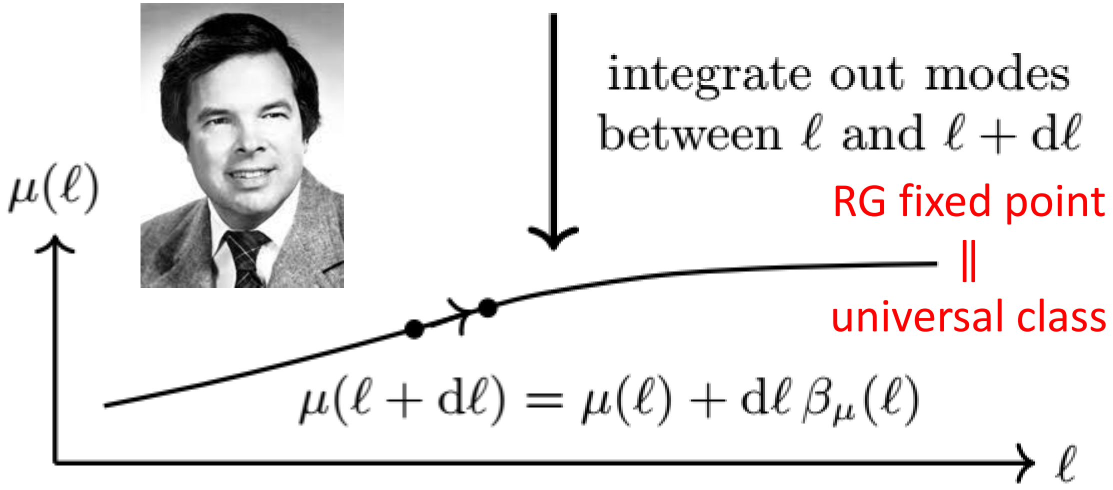
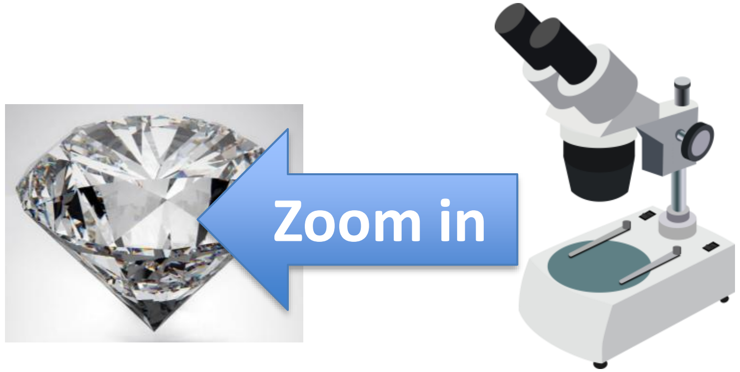


Categorisation: chemical elements vs. universality classes



Group→ ↓Period	1	2		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																		2 He
2	3 Li	4 Be												5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg												13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca		21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr		39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	*	71 Lu	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	**	103 Lr	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Uut	114 Fl	115 Uup	116 Lv	117 Uus	118 Uuo
			*	57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb		
			**	89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No		

Atomic no. → elements

Ising model: $H = \sum_{\langle i,j \rangle} J s_i s_j$

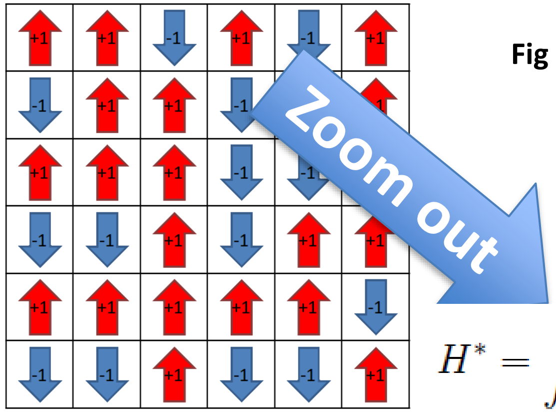


Fig from tbeardsley.com

$$H^* = \int d^d r \left[(\nabla S(\mathbf{r}))^2 + t^* S(\mathbf{r})^2 + u^* S(\mathbf{r})^4 \right]$$

438 Hohenberg and Halperin: Theory of dynamic critical phenomena

TABLE I. Some dynamical models treated by renormalization-group methods.

Model	Designation	System	Dimension order of parameter	Non-conserved fields	Conserved fields	Non-vanishing Poisson bracket
Relaxational	A	Kinetic Ising anisotropic magnets	n	ψ	None	None
	B	Kinetic Ising uniaxial ferromagnet	n	None	ψ	None
	C	Anisotropic magnets structural transition	n	ψ	m	None
Fluid	H	Gas–liquid binary fluid	1	None	ψ $\mathbf{j}$	$\{\psi, \mathbf{j}\}$
Symmetric planar magnet	E	Easy-plane magnet, $h_z = 0$	2	ψ	m	$\{\psi, m\}$
Asymmetric planar magnet	F	Easy-plane magnet, $h_z \neq 0$ superfluid helium	2	ψ	m	$\{\psi, m\}$
Isotropic antiferromagnet	G	Heisenberg antiferromagnet	3	ψ	$\mathbf{m}$	$\{\psi, \mathbf{m}\}$
Isotropic ferromagnet	J	Heisenberg ferromagnet	3	None	ψ	$\{\psi, \psi\}$

Symmetries & conservation laws → UC

 Rev Mod Phys 1977

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4. Diversity of UCs uncovered in the past decades

Two ways to get scale-invariant structures

1. Spontaneous breakage of continuous symmetry: Goldstone modes
2. Fine-tuning to make hydrodynamic field(s) soft: critical & multicritical behaviour

1. Goldstone modes

New nonequilibrium phases

System	Relevant terms	d_c	Exponents	Method	Ref.
Incompressible flocks in 3D	$(\mathbf{v}_T \cdot \nabla)\,\mathbf{v}_T$	4	$z = \frac{8}{5}$, $\zeta = \frac{4}{5}$, $\chi = -\frac{1}{5}$	'Exact (3D)'	Toner & Tu PRL (1995), Chen, Lee, Toner (2018) NJP
Incompressible flocks in 2D	$\lvert\mathbf{v}\rvert^2\mathbf{v}$	$\frac{5}{2}$	$z = \frac{34}{33}$, $\zeta = \frac{2}{3}$, $\chi = -\frac{1}{3}$	1-loop	Chen, Lee, Toner (2016) Nat Comm, Chen, Lee, Toner (2024) PRE

Hydrodynamic EOM of the dominant field:

$$\partial_t \mathbf{u}_\perp + \lambda(\mathbf{u}_\perp \cdot \nabla_\perp)\mathbf{u}_\perp = -\nabla_\perp P + \mu_\perp \nabla_\perp^2 \mathbf{u}_\perp + \mu_x \partial_x^2 \mathbf{u}_\perp + \mathbf{f}_\perp,$$

$$u_n^\perp \partial_n^\perp u_m^\perp = \partial_n^\perp (u_n^\perp u_m^\perp) + (\partial_x \cancel{u_x}) u_m^\perp .$$

Neither μ_x nor noise strength get renormalised

Also, λ not renormalized because of a Galilean symmetry

3 equalities from RG equations → 3 exact scaling exponents

Further support from functional RG analysis:
P. Jentsch and C.F. Lee
Can exact scaling exponents be obtained using the renormalization group? Affirmative evidence from incompressible polar active fluids
E-print: **arXiv:2307.06725**

1. Goldstone modes

New nonequilibrium phases

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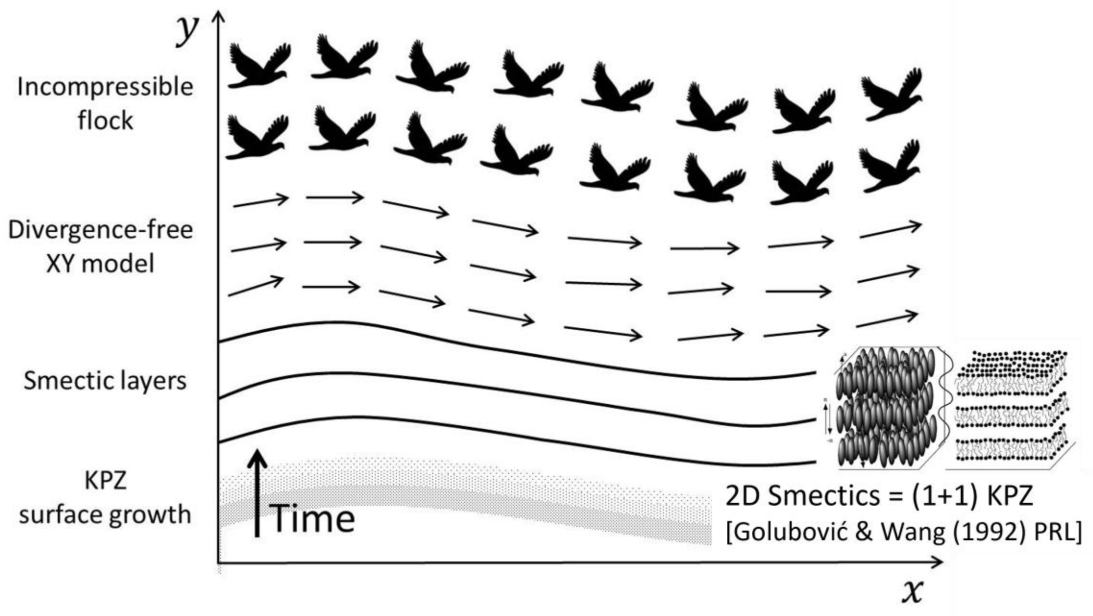

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How about the more generic compressible flocks?

Conflicting results on compressible polar flocks in 2D:

- Analysis unfeasible with current RG methods – Chen, Jentsch, Lee, Maitra, Ramaswamy, Toner (2025) *The inconvenient truth about flocks,* arXiv:2503.17064
- Feasible & scalar exponents can be determined exactly – Chaté & Solon (2024) *Dynamic Scaling of Two-Dimensional Polar Flocks,* PRL 132, 268302

2. Critical & multicritical behaviour

New critical behaviour
(by fine-tuning 2 parameters)

System	Relevant terms	d_c	Exponents	Method	Ref.
In the ordered phase	$\nabla_\perp \delta\rho^3$	5	$z=2\ ,\ \chi=-\frac{2-\epsilon}{3}$	1-loop	Miller & Toner (2024) PRE
With an easy axis	$\partial_x g_x^2\ ,\ g_x^3\ ,\ \partial_x \delta\rho^2\ ,\ \delta\rho^2 g_x$	4	$z=2\ ,\ \chi=-\frac{2-\epsilon}{2}$	1-loop	Wong & Lee (2025) arXiv
Incompressible limit	$(\mathbf{v}\cdot\nabla)\,\mathbf{v}\ ,\ \lvert\mathbf{v}\rvert^2\,\mathbf{v}$	4	$z=2-\frac{31\epsilon}{113}\ ,\ \chi=-1+\frac{41\epsilon}{113}$	1-loop	Chen, Toner, Lee (2015) NJP
Incompressible limit, with an additional spin field s	$\lvert\mathbf{v}\rvert^2\,\mathbf{v}\ ,\ (\mathbf{v}\cdot\nabla)\,\mathbf{v}\ ,\ (\mathbf{v}\cdot\nabla)\,s\ ,\ \mathbf{v}\cdot s\ ,\ \mathbf{v}\nabla^2\mathbf{v}$	4	$z=2-0.65(2)\epsilon$, χ not calculated	1-loop	Cavagna *et al.* (2023) Nat. Phys.

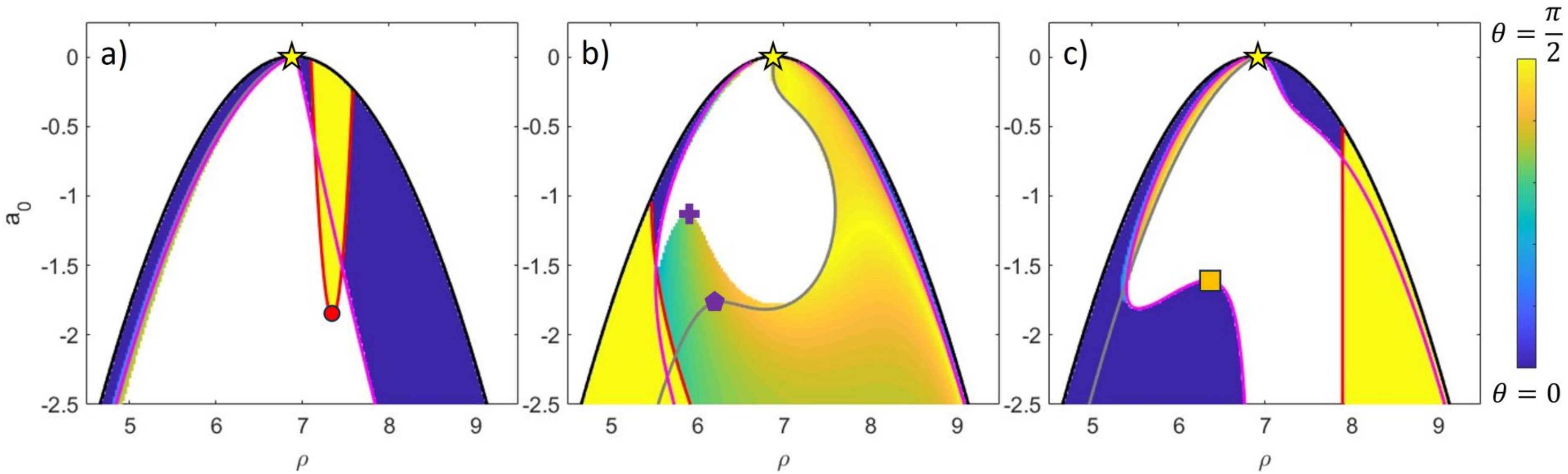
New Multi-critical behaviour
(by fine-tuning >2 parameters)

System	Relevant terms	d_c	# of new UCs	Method	Ref.
With an easy axis	$v_x^3\ ,\ \delta\rho^2 v_x$	4	2	1-loop	Wong & Lee (2025) arXiv
Ordered phase, with omitted terms	$(\mathbf{v}_\perp\cdot\nabla_\perp)\mathbf{v}_\perp\ ,\ \mathbf{v}_\perp(\nabla_\perp\cdot\mathbf{v}_\perp)\ ,\ \nabla_\perp\lvert\mathbf{v}_\perp\rvert^2$	$\frac{11}{3}$	1	Nonperturbative	Jentsch & Lee (2024) PRL
Multiple fine-tuning	$\delta\rho^2\mathbf{v}\ ,\ \delta\rho^2\nabla\delta\rho$	6	3	Nonperturbative	Jentsch & Lee (2023) PRR

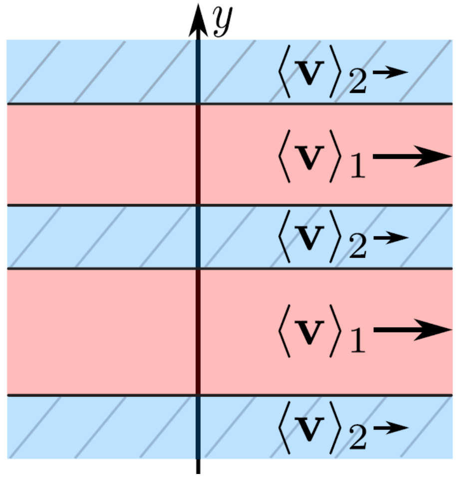
Jentsch & Lee, review in preparation

Newly predicted critical behaviour in the ordered phase

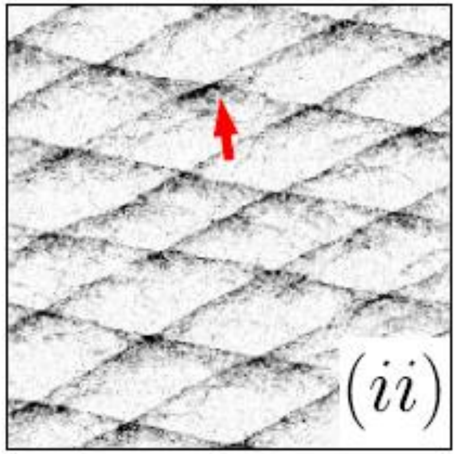
SIS via fine-tuning:
Not the usual critical order-disorder transition,
but critical phase separation/instability



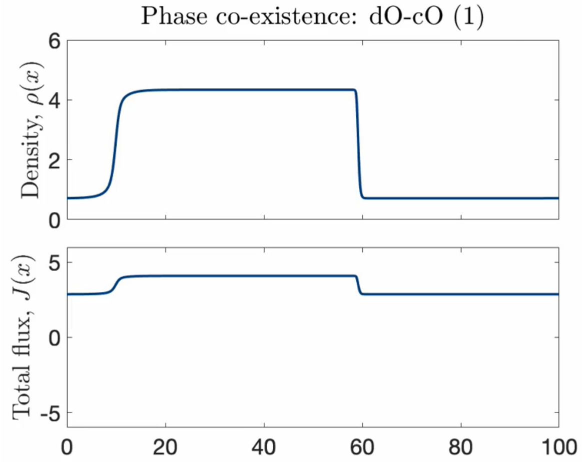
P. Jentsch and C.F. Lee (2025)
Diversity of critical phenomena in the ordered phase of polar active fluids.
E-print: arXiv:2512.18846



Miller & Toner (2024) PRE



Kürsten & Ihle (2020) PRL



Bertrand & CFL (2022) PRR

Summary

- Large natural and engineered swarms can reach scales where scale-invariant structures (SIS) emerge, motivating a hydrodynamic description of long-wavelength, long-time dynamics
- Symmetry principles and conservation laws determine the universal hydrodynamic equations of motion
- SIS arise through two generic routes
  - spontaneous breaking of continuous symmetry
  - fine-tuning that softens hydrodynamic modes
- Over the past decade, this framework has revealed a diverse landscape of universality classes in swarming systems
- Multiple new critical behaviours are predicted within the ordered phase, many still awaiting full RG treatment

Outlook

- **Grand challenge 1:** What is the UC that governs **generic compressible polar flocks?**

- **Grand challenge 2:** Can we complete the periodic table of nonequilibrium universality classes for adaptive swarms?

Acknowledgement

Patrick Jentsch (EMBL)

Thank you for your attention

A PhD position is available to start in Autumn 2026, open to all nationalities

Details on clee.bg-research.cc.ic.ac.uk/index.html, also on findaphd.com

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