DAMTP Statistical Physics and Soft Matter Seminar
Cambridge, 12 November 2024

Where to dig for gold?

Simple ‘tricks’ to help find new universality classes

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Physics $\approx$ Universal behaviour + O(model specific details) *

*Asymptotic series: leading terms can be different

Mathematical definition of universality

Universality Class = Fixed point (**FP**) of renormalisation group (**RG**) coarse graining procedure

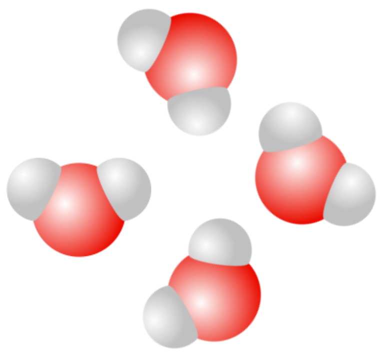
Universal behaviour = Properties quantifiable by RG FP

Relevant terms in model equations = Terms that affect RG FP

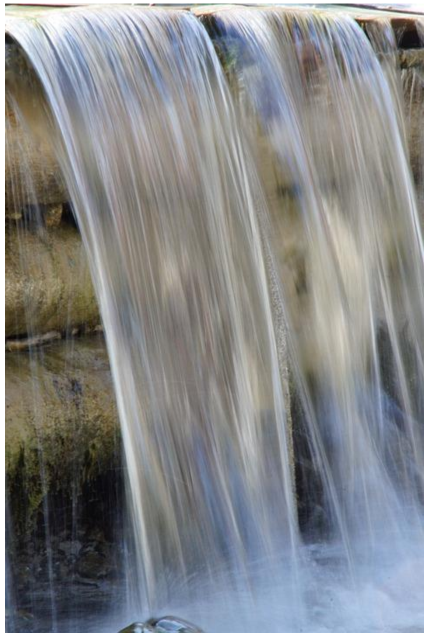
Irrelevant terms = Terms that do not affect RG FP

Renormalisation group

A mathematical, coarse-graining procedure that incorporates fluctuations to quantify how model parameters are modified at larger length scales



RG transformation →



Categorisation: chemical elements vs. universality classes

Group→ ↓Period	1	2		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																		2 He
2	3 Li	4 Be												5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg												13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca		21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr		39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	*	71 Lu	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	**	103 Lr	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Uut	114 Fl	115 Uup	116 Lv	117 Uus	118 Uuo
			*	57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb		
			**	89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No		

Atomic no. → elements

Ising model: $H = \sum_{\langle i,j \rangle} J s_i s_j$

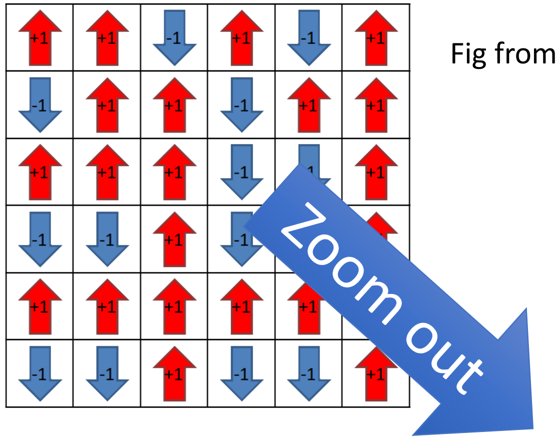
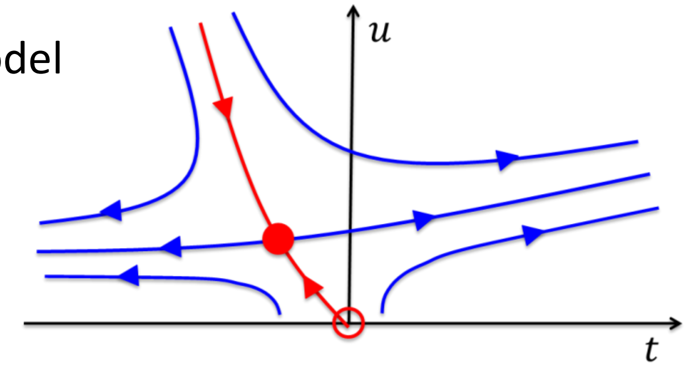


Fig from tbeardsley.com

$$H^* = \int d^d r \left[(\nabla S(\mathbf{r}))^2 + t^* S(\mathbf{r})^2 + u^* S(\mathbf{r})^4 \right]$$

RG flow of model parameters



Symmetries & conservation laws (SCL) → UC

Why universality classes ≈ gold?

- UCs are eternal
- Its utility tends to go up with time
  - Kardar-Parisi-Zhang model as an example

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Dynamic Scaling of Growing Interfaces

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ARTICLE

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Mapping two-dimensional polar active fluids to two-dimensional soap and one-dimensional sandblasting

Leiming Chen[1], Chiu Fan Lee[2] & John Toner[3,4]

PHYSICAL REVIEW LETTERS **128,** 070401 (2022)

Emergent Kardar-Parisi-Zhang Phase in Quadratically Driven Condensates

Oriana K. Diessel,[1] Sebastian Diehl,[2] and Alessio Chiocchetta[2]
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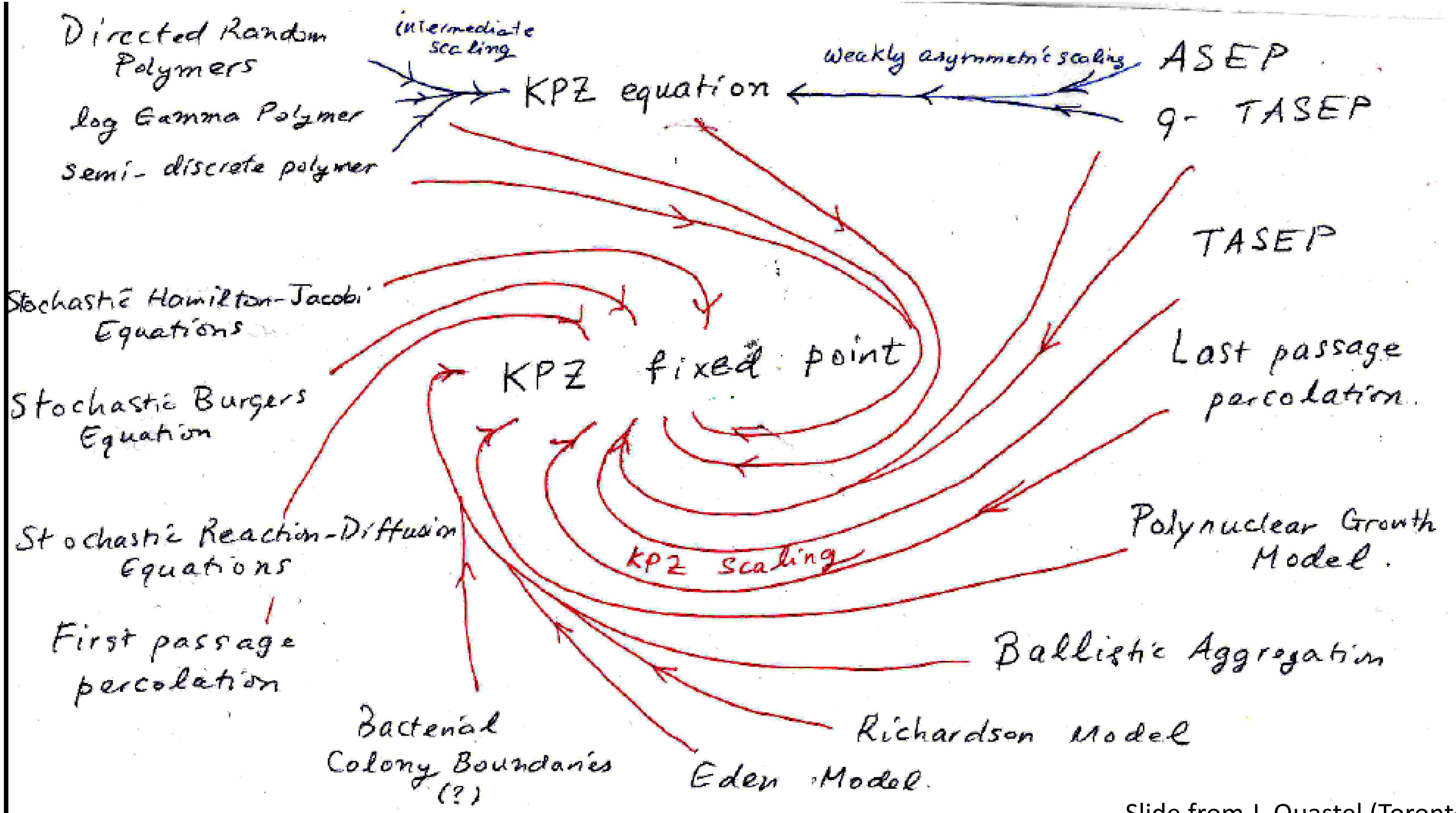
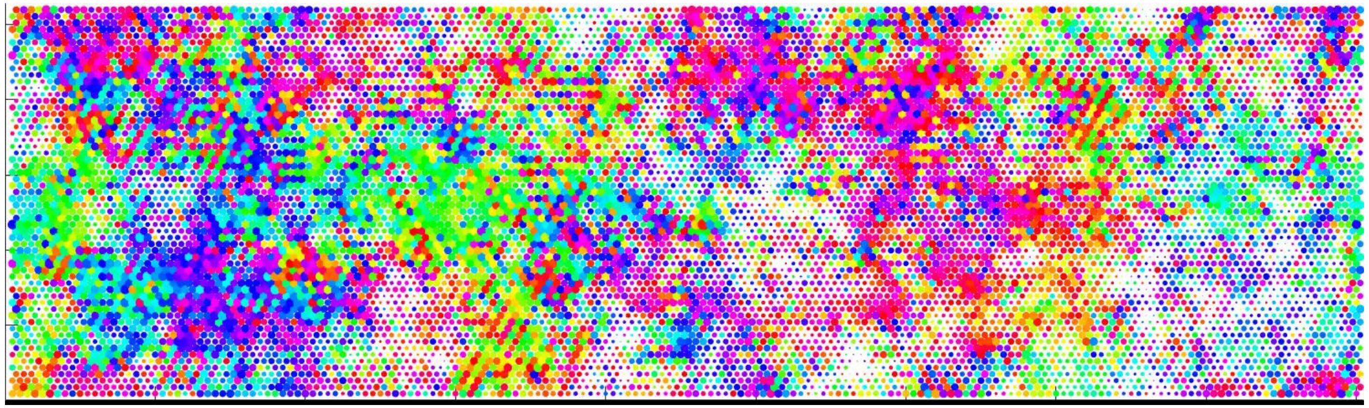


Slide from J. Quastel (Toronto)

Simple tricks to identify new UCs

1. Start with a model with new symmetries & conservation laws
2. Mean-field analysis
3. Linear stability analysis
4. Linear fluctuating hydrodynamics
5. Power counting (zeroth order RG) analysis

Colour wheel used to determine direction of velocity



D Nesbitt, G Pruessner CFL (2021) NJP 23, 043047

1. Start with a model with new symmetries & conservation laws: *Polar active fluids as an example*

Polar active fluids

- Hydrodynamic variables: density ρ and momentum $\vec{g}$

- Conservation law: Mass conservation

$$\boxed{\begin{aligned} \partial_t \rho + \nabla \cdot \mathbf{g} &= 0 \\ \partial_t \mathbf{g} &= \mathbf{F} \end{aligned}}$$

Symmetries

$$\begin{aligned} \partial_t \rho + \nabla \cdot \mathbf{g} &= 0 \\ \partial_t \mathbf{g} &= \mathbf{F} \end{aligned}$$

- What is the force **F**?

- Starting with symmetries:
  - Temporal invariance: **F** does not depend on time
  - Translational invariance: **F** does not depend on position **r**
  - Rotational invariance: **F** does not depend on a particular direction
  - Chiral (parity) invariance: **F** is not right-handed or left-handed

Toner-Tu Equations of Motion (EOM)

Gaussian noise terms

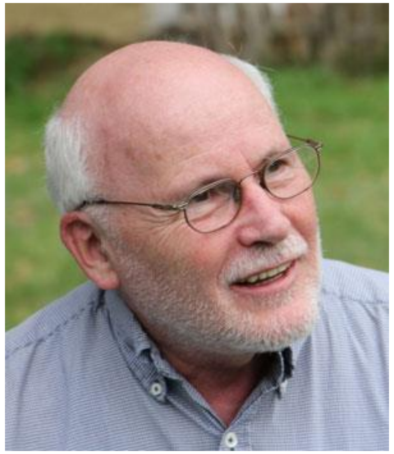
$$\langle f(\mathbf{r},t)\rangle = 0$$

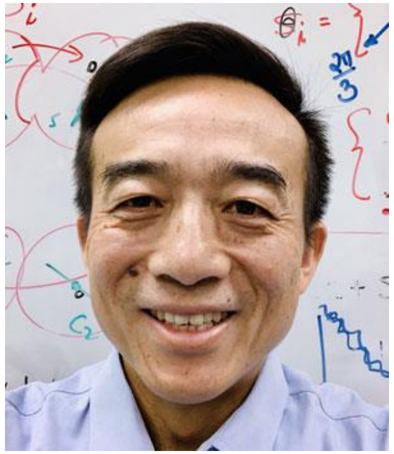
$$\langle f(\mathbf{r},t)f(\mathbf{r}',t')\rangle = 2D\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

$$\partial_t \rho + \nabla\cdot\mathbf{g} = 0$$

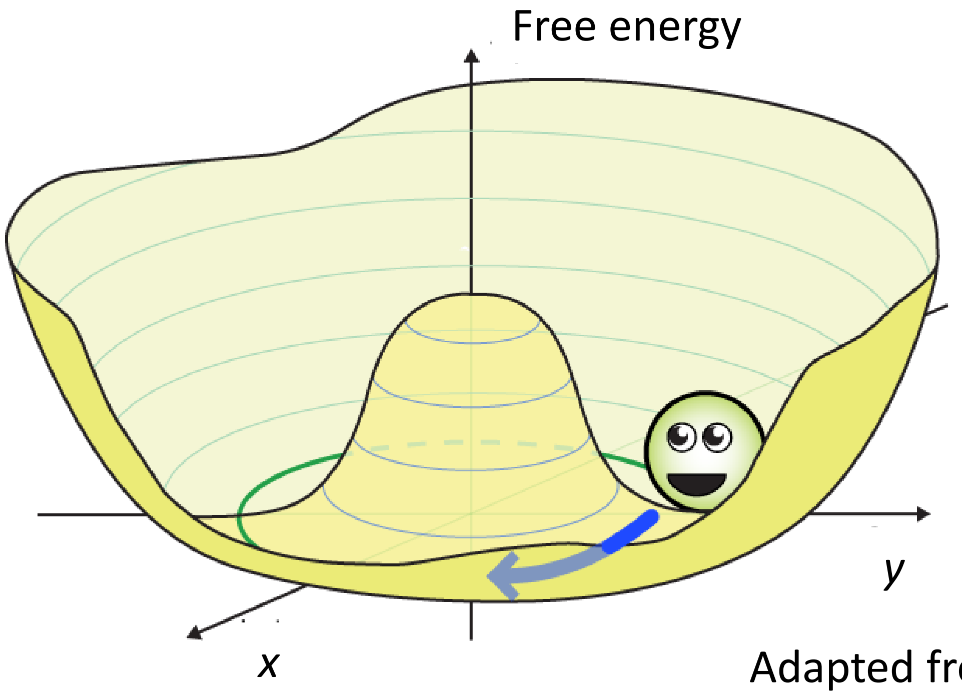
$$\partial_t \mathbf{g} + \lambda_1(\mathbf{g}\cdot\nabla)\mathbf{g} + \lambda_2(\nabla\cdot\mathbf{g})\mathbf{g} + \lambda_3\nabla g^2 = U(\rho,\mathbf{g})\mathbf{g} - \kappa_1\nabla\rho - \kappa_2\mathbf{g}(\mathbf{g}\cdot\nabla\rho) + \mu\nabla^2\mathbf{g} + \mathbf{f} + \cdots$$

2020 APS Onsager Prize





Citation: "*For seminal work on the theory of flocking that marked the birth and contributed greatly to the development of the field of active matter.*"



Adapted from QuantumDiaries.org

2. Mean-field analysis

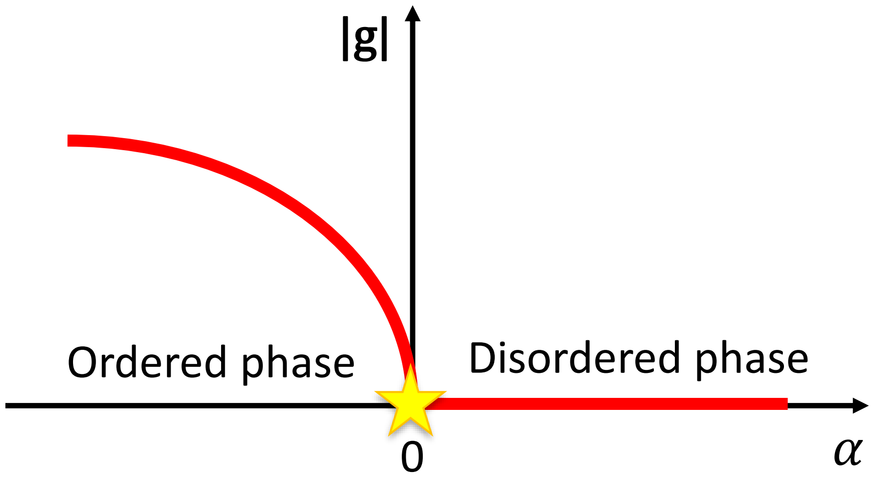
Mean-field analysis of polar active fluids

$$\partial_t \rho + \nabla \cdot \mathbf{g} = 0$$

$$\partial_t \mathbf{g} + \lambda_1 (\mathbf{g}\cdot\nabla)\mathbf{g} + \lambda_2 (\nabla\cdot\mathbf{g})\mathbf{g} + \lambda_3 \nabla g^2 = U(\rho, \mathbf{g})\mathbf{g} - \kappa_1 \nabla \rho - \kappa_2 \mathbf{g}(\mathbf{g}\cdot\nabla\rho) + \mu \nabla^2 \mathbf{g} + \mathbf{f} + \cdots$$

$$-(\alpha + \beta g^2 + \cdots)\mathbf{g}$$

Spatiotemporally homogeneous solutions:

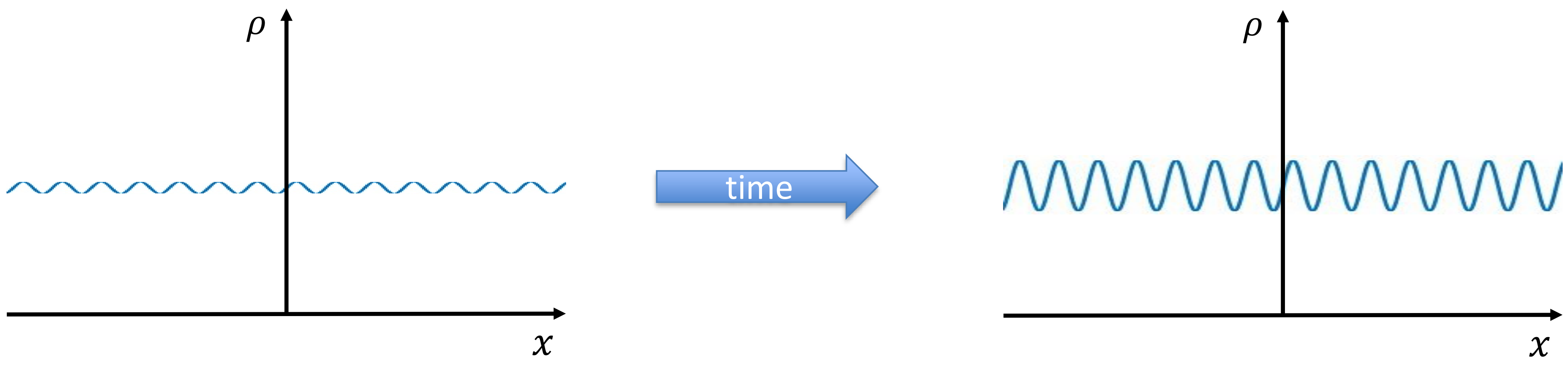


1. Ordered phase of systems with new SCL
→ New UC for the ordered phase?

2. Critical transition from disordered phase to a new ordered phase
→ New UC for the critical behaviour?

Focus of today

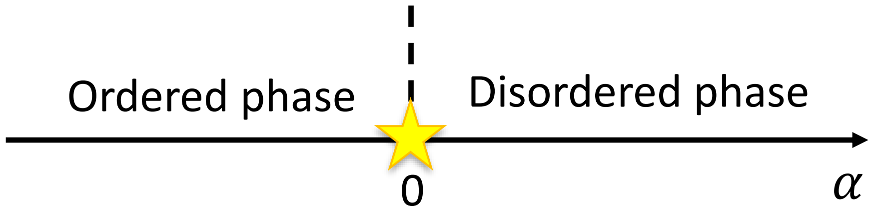
MF $\rightarrow$ A new phase and a new critical UC



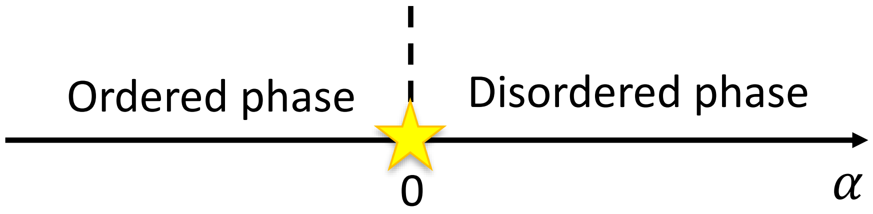
3. Linear stability analysis

Linear stability analysis

- MF analysis → two spatiotemporally homogeneous phases



Linear stability: disordered phase



- Start with linearised TT EOM: $\partial_t \rho + \nabla \cdot \mathbf{g} = 0\ , \qquad \partial_t \mathbf{g} = -\alpha_0 \mathbf{g} - \kappa_1 \nabla \delta\rho + \mu_1 \nabla^2 \mathbf{g}$

- Consider homogeneous state with perturbations:
$$\begin{aligned} \rho(t,\mathbf{r}) &= \rho_0 + \epsilon_\rho \exp\left(st - \mathrm{i}\mathbf{q}\cdot\mathbf{r}\right) \\ \vec{g}(t,\mathbf{r}) &= \vec{\epsilon}_g \exp\left(st - \mathrm{i}\mathbf{q}\cdot\mathbf{r}\right) \end{aligned}$$

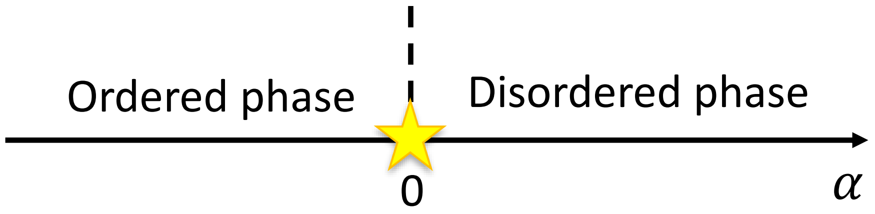
- Substitution into linearised EOM: $s\epsilon_\rho + \mathrm{i}\mathbf{q}\cdot\vec{\epsilon}_g = 0\ , \qquad s\vec{\epsilon}_g = -\alpha_0\vec{\epsilon}_g - \kappa_1 \mathrm{i}\mathbf{q}\epsilon_\rho - \mu_1 q^2 \vec{\epsilon}_g$

- Re-write as an eigenvalue problem: $s\Phi = M_D \cdot \Phi\ , \qquad \Phi = \left(\epsilon_\rho \ \ \vec{\epsilon}_g\right)^T$
$$M_D = \begin{pmatrix} 0 & -\mathrm{i}q & 0 \\ -\mathrm{i}\kappa_1 q & -\alpha_0 - \mu q^2 & 0 \\ 0 & 0 & \left(-\alpha_0 - \mu q^2\right)\mathbf{I}_{d-1} \end{pmatrix}$$

- Solve for eigenvalues:
$$s = \begin{cases} -\alpha_0 - \mu q^2 \\ -\frac{\alpha_0 + \mu q^2}{2} \pm \sqrt{\frac{(\alpha_0+\mu q^2)^2}{4} - \kappa_1 q^2} \end{cases} \xrightarrow{q\to 0} \begin{cases} -\alpha_0 \\ -\frac{\kappa_1}{\alpha_0} q^2 \end{cases}$$

Always negative!

Linear stability: disordered phase



- Start with linearised TT EOM: $\partial_t \rho + \nabla \cdot \mathbf{g} = 0\ , \qquad \partial_t \mathbf{g} = -\alpha_0 \mathbf{g} - \kappa_1 \nabla \delta\rho + \mu_1 \nabla^2 \mathbf{g}$

- Consider homogeneous state with perturbations:

$$\begin{aligned} \rho(t,\mathbf{r}) &= \rho_0 + \epsilon_\rho \exp\left(st - i\mathbf{q}\cdot\mathbf{r}\right) \\ \vec{g}(t,\mathbf{r}) &= \vec{\epsilon}_g \exp\left(st - i\mathbf{q}\cdot\mathbf{r}\right) \end{aligned}$$

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- Re-write as an eigenvalue problem: $s\Phi = M_D \cdot \Phi\ , \qquad \Phi = \left(\epsilon_\rho \ \ \vec{\epsilon}_g\right)^T$

$$M_D = \begin{pmatrix} 0 & -i\dots \\ -i\dots & (-\alpha_0 - \mu q^2)\,\mathbf{I}_{d-1} \end{pmatrix}$$

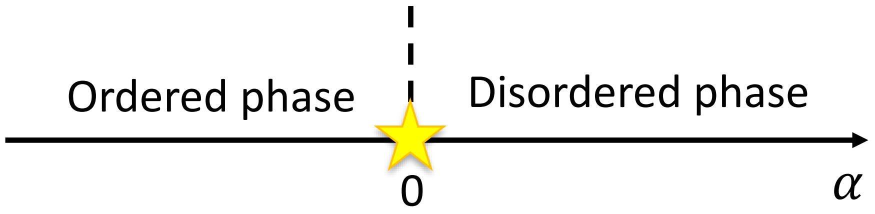
- Solve for eigenvalues:

$$s = \begin{cases} \dots \\ -\frac{\alpha_0+\mu q^2}{2} \pm \sqrt{\frac{(\alpha_0+\mu q^2)^2}{4} - \kappa_1 q^2} \end{cases} \xrightarrow{q\to 0} \begin{cases} -\alpha_0 \\ -\frac{\kappa_1}{\alpha_0} q^2 \end{cases}$$

Always negative!

Disordered phase is always linearly stable!

Linear stability: Ordered phase



- Start with linearised TT EOM:

$$\begin{aligned} \partial_t \rho + \nabla \cdot \mathbf{g} &= 0 \\ \partial_t \mathbf{g} + \lambda \mathbf{g} \cdot \nabla \mathbf{g} &= \mu \nabla^2 \mathbf{g} - \kappa \nabla \rho + \alpha(\rho)\mathbf{g} - \beta g^2 \mathbf{g} \end{aligned}$$

- Consider homogeneous state with perturbations:

$$\begin{aligned} \rho(t, \mathbf{r}) &= \rho_0 + \epsilon_\rho \exp\left(st - \mathrm{i}\mathbf{q} \cdot \mathbf{r}\right) \\ \vec{g}(t, \mathbf{r}) &= g_0 \hat{\mathbf{x}} + \vec{\epsilon}_g \exp\left(st - \mathrm{i}\mathbf{q} \cdot \mathbf{r}\right) \end{aligned}$$

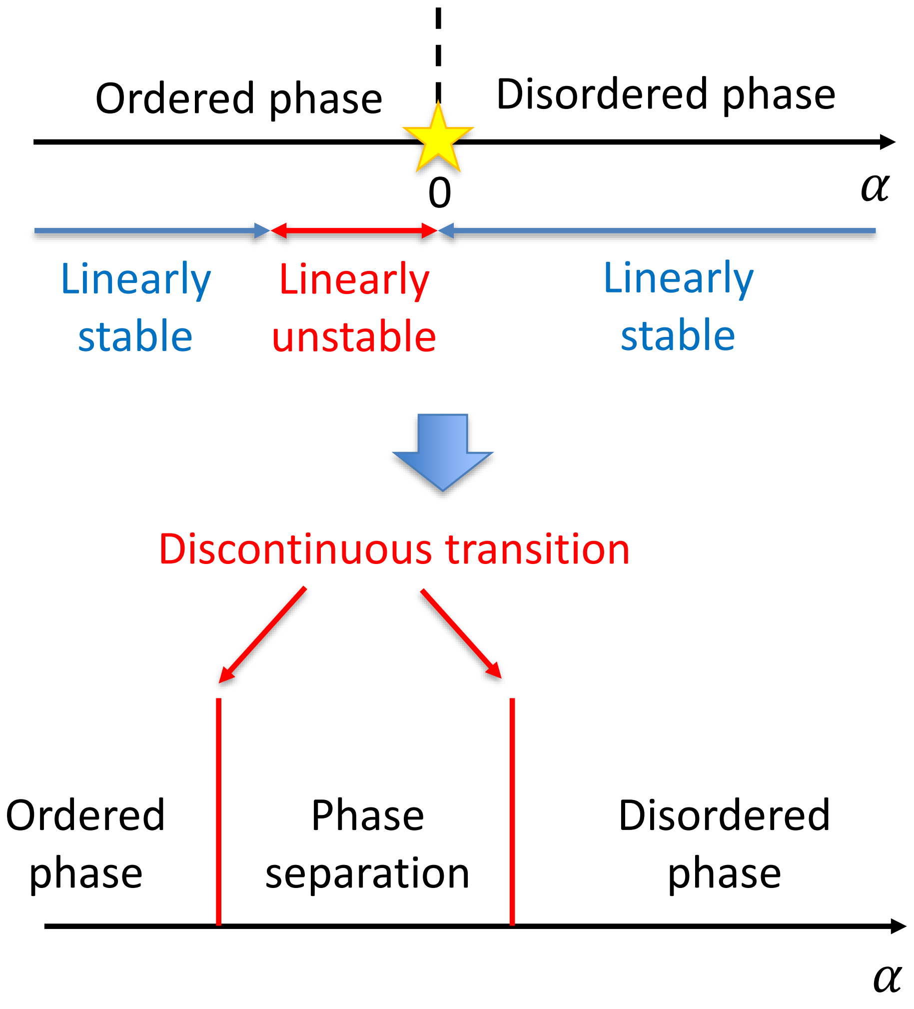
- Substitution into linearised EOM

- Re-write as an eigenvalue problem

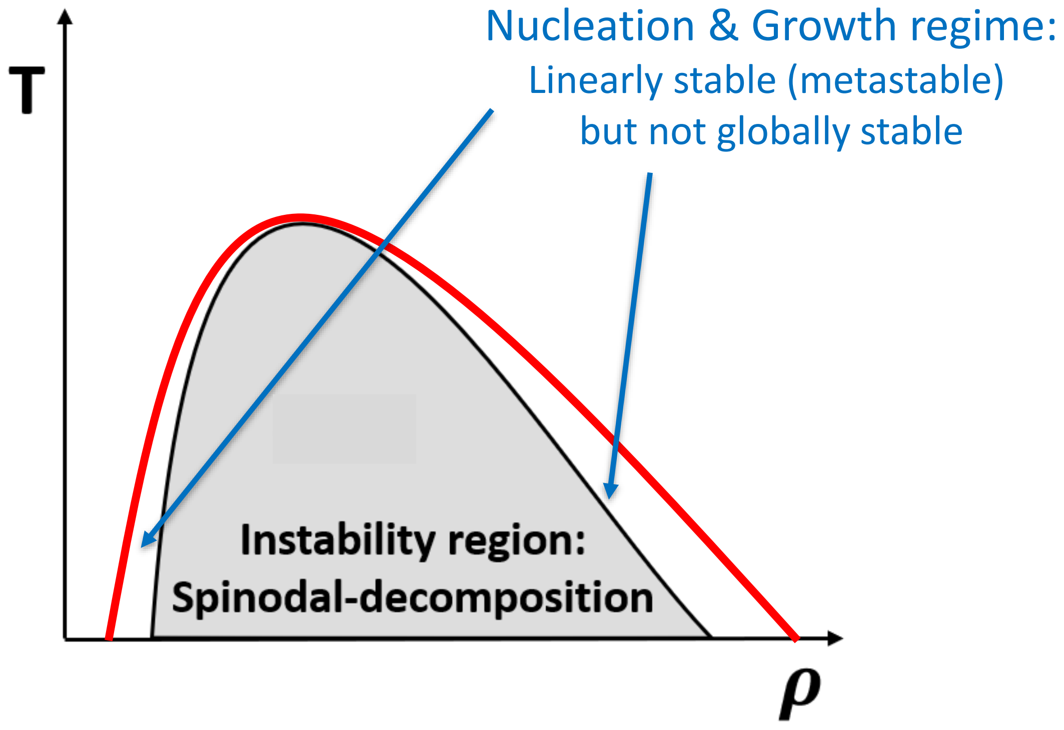
- Solve for eigenvalues: $s = \left[\frac{\alpha_1^2}{\beta} - 2|\alpha_0|\left(2\kappa + \frac{\alpha_1 \lambda}{\beta}\right)\right] q^2 + \mathcal{O}(q^4)$ Always positive when α_0 is small!

where $\alpha(\rho_0 + \delta\rho) = \alpha_0 + \alpha_1 \delta\rho + \mathcal{O}(\delta\rho^2)$

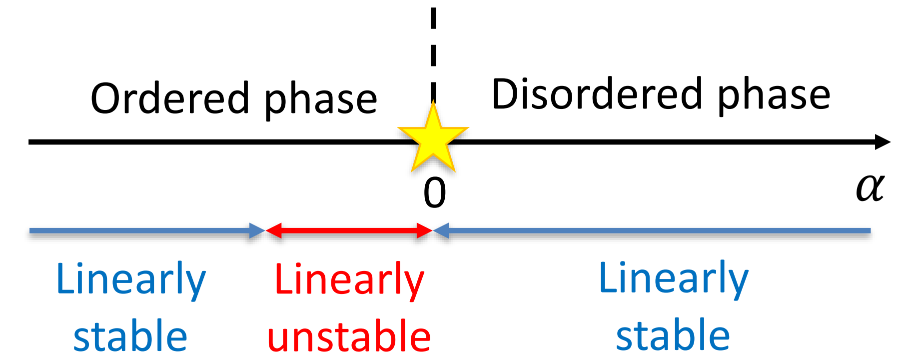
Linear stability analysis → more complicated phase diagram



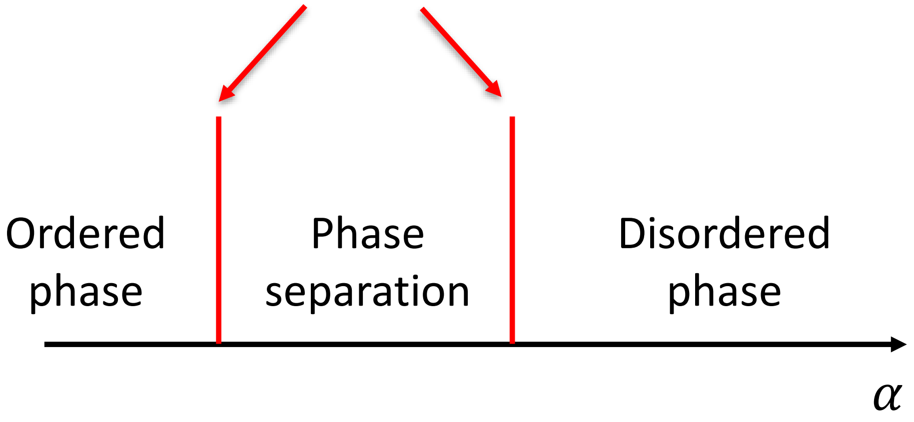
Similar to thermal phase separation:



Linear stability analysis → more complicated phase diagram



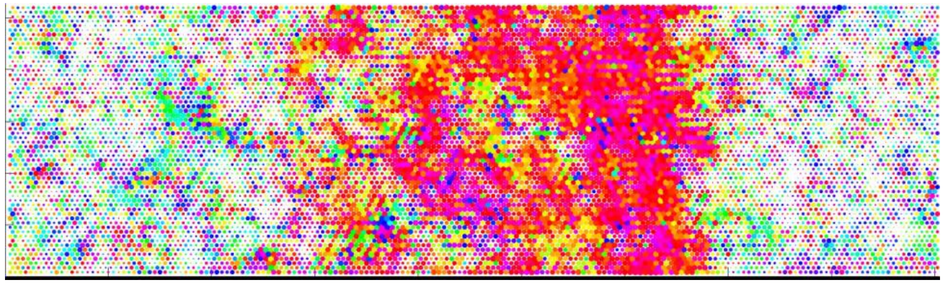
Discontinuous transition



Phase separation → Banding regime

G Grégoire & H Chaté (2004) PRL 92, 025702
E Bertin, M Droz & G Grégoire (2006) PRE 74, 022101

Colour wheel used to determine direction of velocity 



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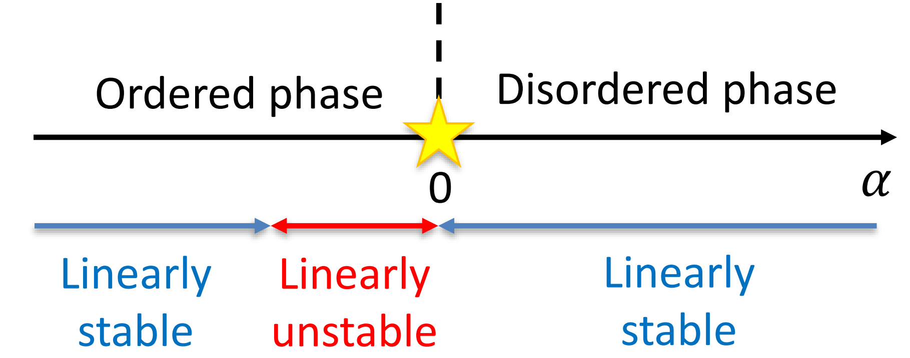
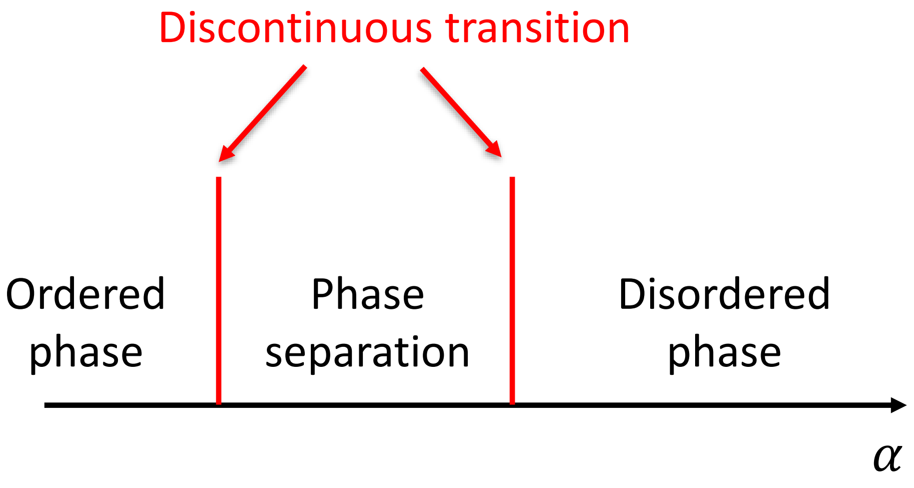
Linear stability analysis → more complicated phase diagram





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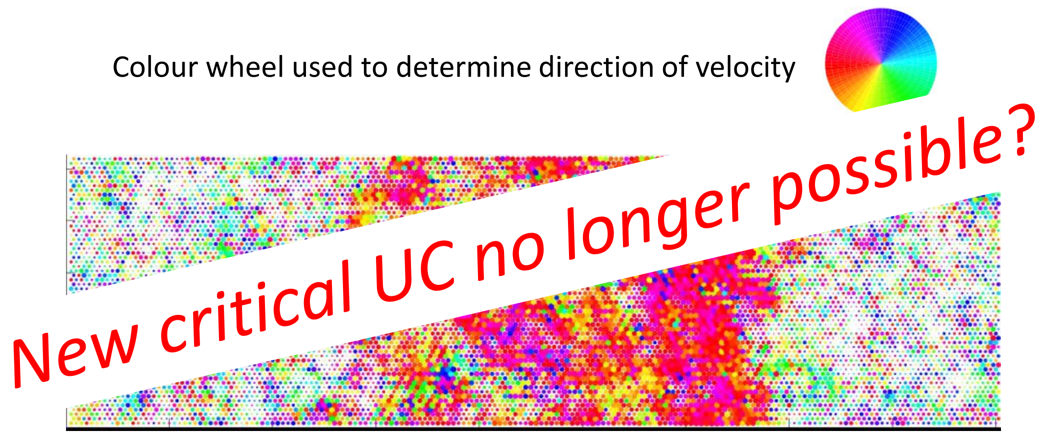

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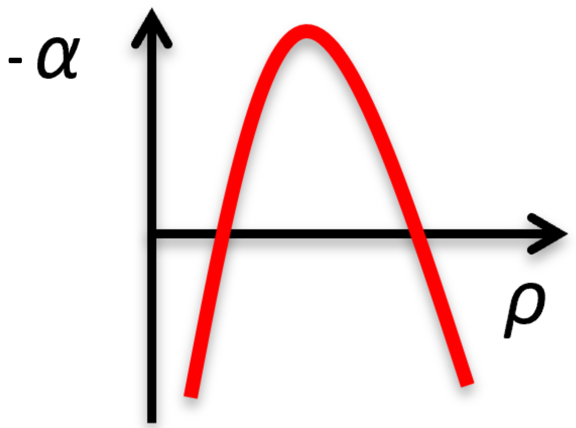
No so fast!

Remember that in the ordered phase, the key eigenvalue is $s = \left[\frac{\alpha_1^2}{\beta} - 2|\alpha_0|\left(2\kappa + \frac{\alpha_1\lambda}{\beta}\right)\right] q^2 + \mathcal{O}(q^4)$

where $\alpha(\rho_0 + \delta\rho) = \alpha_0 + \alpha_1 \delta\rho + \mathcal{O}(\delta\rho^2)$

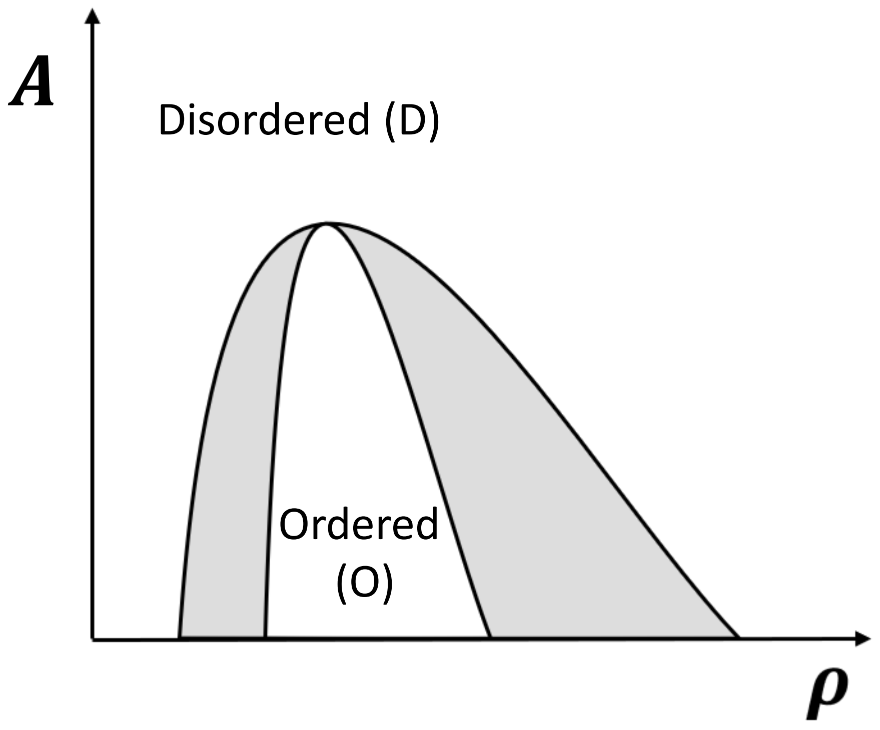
What if α_1 is zero? E.g., collective motion speed goes down with density :



→ α_1 can be zero and so no instability as all!

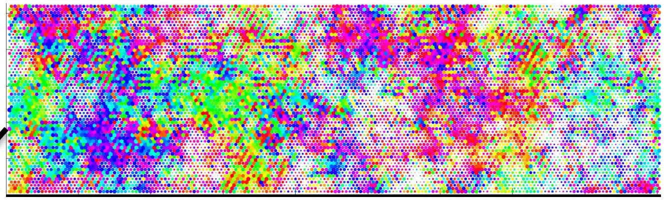
An example

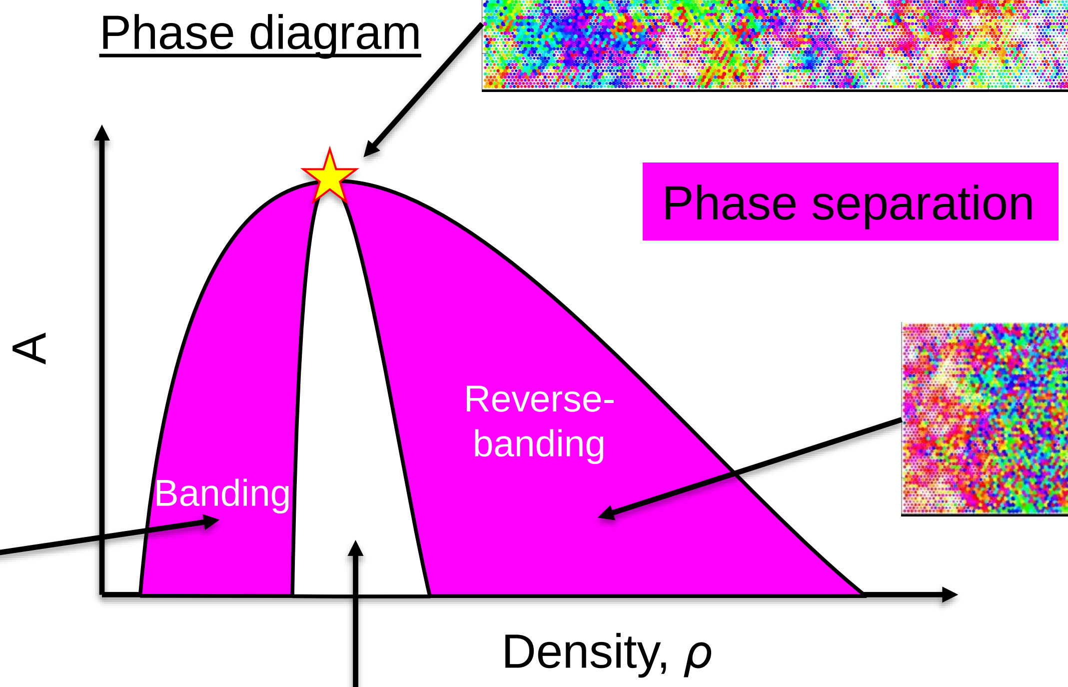
Ex: $-\alpha(\rho) = -A + \rho - \rho^2$

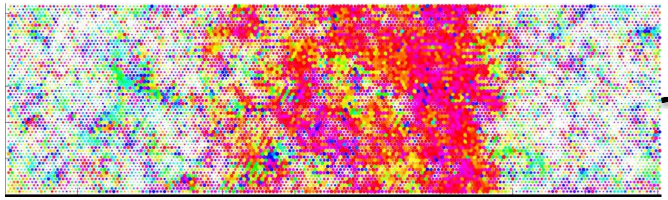


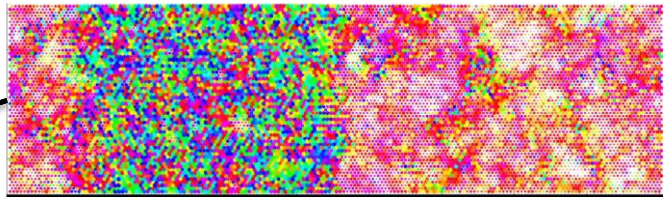
Recovery of ordered-disordered critical point

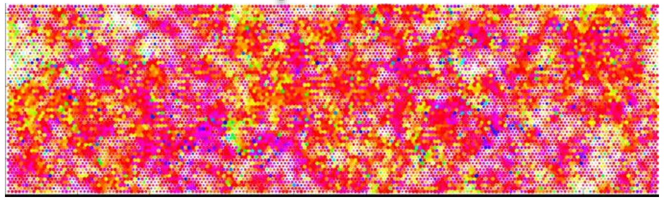
Colour wheel used to determine direction of velocity







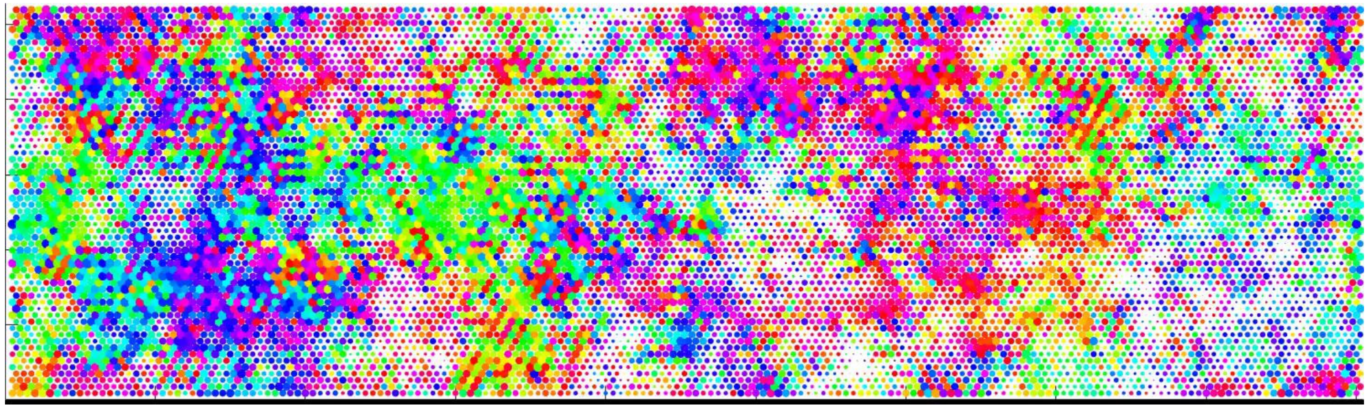




D Nesbitt, G Pruessner & CFL (2021) NJP 23, 043047;
See also T Agranov et al (2024) NJP 26, 063006

Linear stability → An enriched phase diagram with a new critical UC by fine tuning 2 parameters

Colour wheel used to determine direction of velocity



D Nesbitt, G Pruessner CFL (2021) NJP 23, 043047

4. Linear fluctuating hydrodynamics

Linear fluctuating hydrodynamics at criticality

- Start with linearised TT EOM with noise:

$$\partial_t \rho + \nabla \cdot \mathbf{g} = 0\ , \qquad \partial_t \mathbf{g} = \cancel{-\alpha_0 \mathbf{g}} - \kappa_1 \nabla \delta\rho + \mu_1 \nabla^2 \mathbf{g} + f$$

- Fourier transformed spatiotemporally

$$\delta\rho(\omega_q, \mathbf{q}) = \int_{\tilde{\mathbf{r}}} e^{\mathrm{i}\omega t - \mathrm{i}\mathbf{q}\cdot\mathbf{r}} \delta\rho(t, \mathbf{r})\ ,$$

$$\mathbf{g}(\omega_q, \mathbf{q}) = \int_{\tilde{\mathbf{r}}} e^{\mathrm{i}\omega t - \mathrm{i}\mathbf{q}\cdot\mathbf{r}} \mathbf{g}(t, \mathbf{r})$$

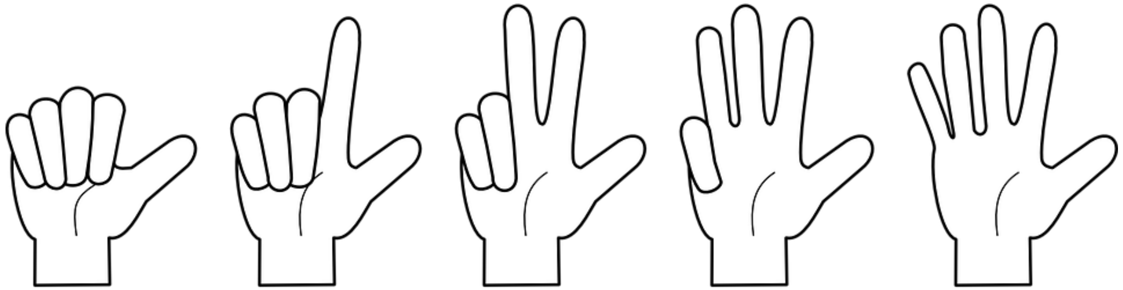
- Re-write in matrix form

- Find correlation functions between hydrodynamic fields to find

$$\langle \rho(\mathbf{r}, t)\rho(\mathbf{r}', t)\rangle \propto |\mathbf{r} - \mathbf{r}'|^{2-d} \qquad \langle \mathbf{g}(\mathbf{r}, t)\cdot \mathbf{g}(\mathbf{r}', t)\rangle \propto |\mathbf{r} - \mathbf{r}'|^{2-d}$$

$$\langle \rho(\mathbf{r}, t)\rho(\mathbf{r}, t')\rangle \propto |t - t'|^{(2-d)/2} \qquad \langle \mathbf{g}(\mathbf{r}, t)\cdot \mathbf{g}(\mathbf{r}, t')\rangle \propto |t - t'|^{(2-d)/2}$$

Linear fluctuating hydrodynamics → power-law correlations among hydrodynamic fields at criticality



5. Power counting (zeroth order RG) analysis

Is it a new UC?

$$\mathbf{r} \mapsto \mathrm{e}^{\ell}\ ,\ t \mapsto \mathrm{e}^{z\ell} t\ ,\ \delta\rho \mapsto \mathrm{e}^{\chi\ell}\delta\rho\ ,\ \mathbf{g} \mapsto \mathrm{e}^{\chi\ell}\mathbf{g}$$

- Remember the scaling exponents from the linear theory: $z = 2\ ,\ \chi = \frac{2-d}{2}$
- Use them to ascertain 1) the upper critical dimension, and 2) relevant nonlinear terms
- Here, upper critical dimension is 4, and relevant terms are

$$\partial_t \mathbf{g} + \lambda_1(\mathbf{g}\cdot\nabla)\mathbf{g} + \lambda_2\mathbf{g}(\nabla\cdot\mathbf{g}) + \frac{\lambda_3}{2}\nabla(|\mathbf{g}|^2) = -\beta|\mathbf{g}|^2\mathbf{g} + \cdots + \mathbf{f}$$

- No previous models have studied these nonlinearities before!

Power counting → critical model contains nonlinearities never studied before → indicative of a new UC

Summary: simple tricks to identify new UCs

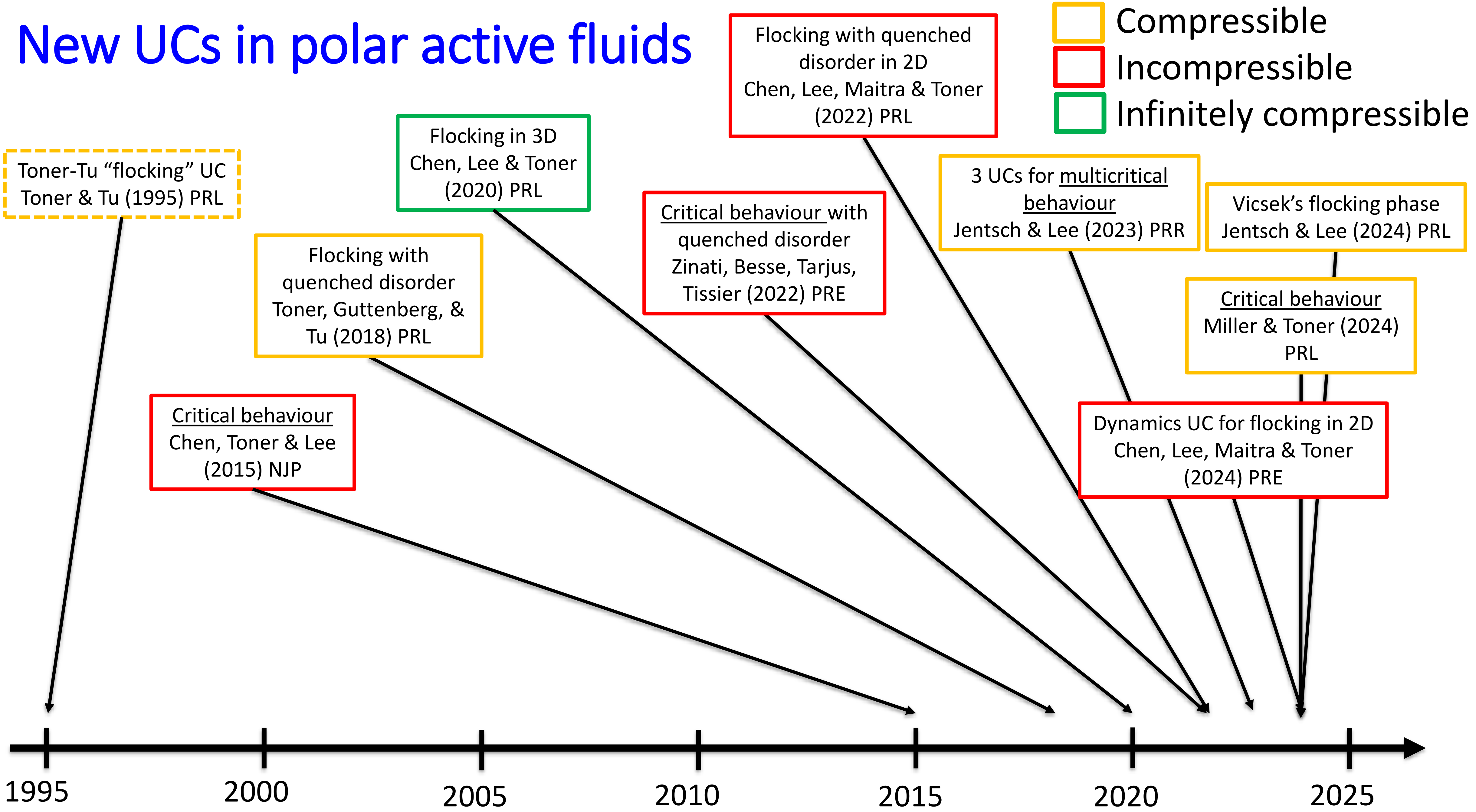
1. New set of symmetries & conservation laws: polar active fluids as an example
2. Mean-field analysis → critical point
3. Linear stability analysis → enriched phase diagram
4. Linear fluctuating hydrodynamics → critical exponents
5. Power counting (zeroth order RG) analysis → upper critical dimensions + relevant terms

Outlook

Now that we know where to dig, how do we actually start digging?

Gold standard: Renormalisation group analysis (Wilsonian, field-theoretic, functional)

New UCs in polar active fluids



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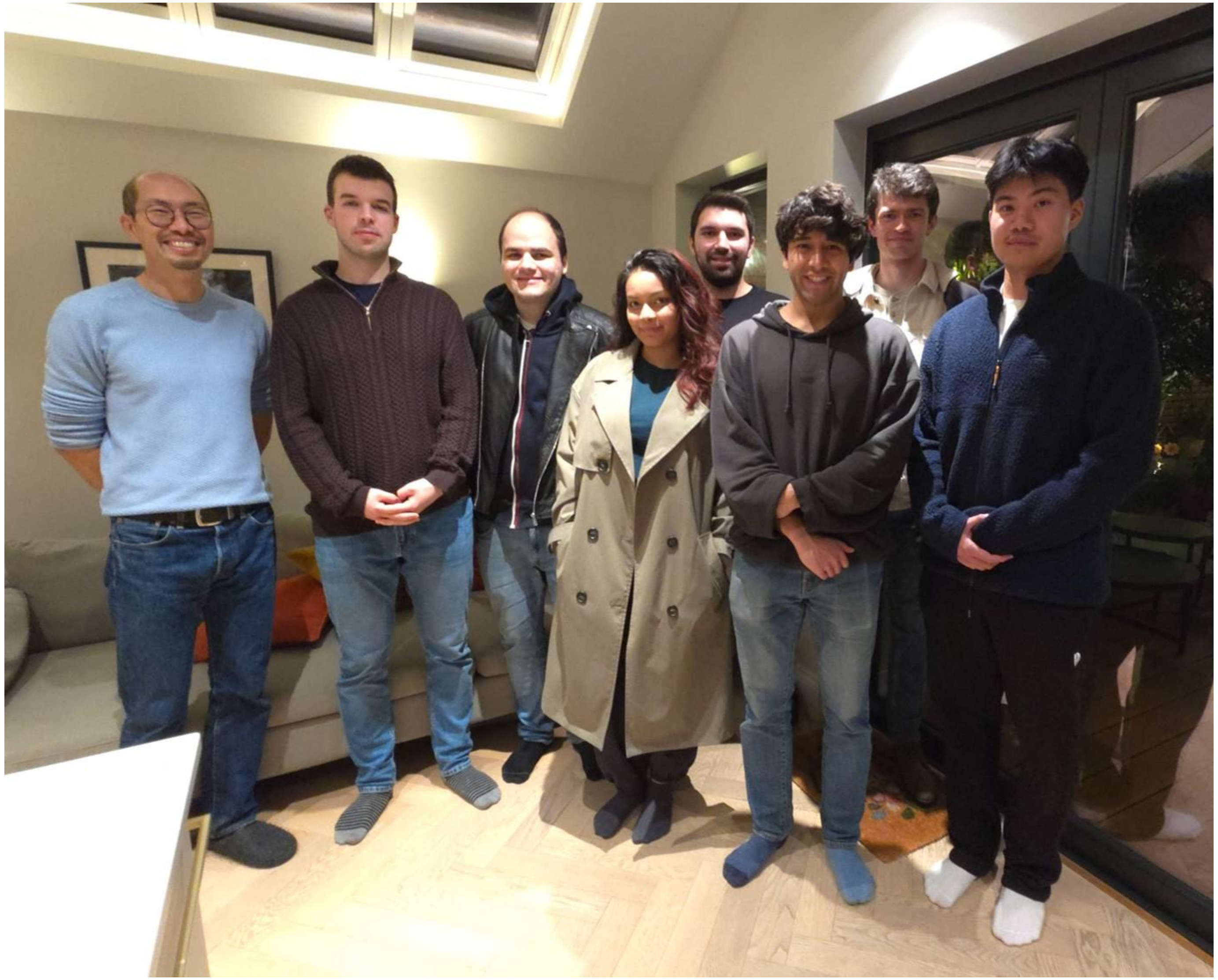
Thibault Bertrand
Mathematics, Imperial College



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